

# INCOMPRESSIBLE LIMIT OF THE COMPRESSIBLE NON-ISENTROPIC MAGNETOHYDRODYNAMIC EQUATIONS WITH ZERO MAGNETIC DIFFUSIVITY

SONG JIANG, QIANGCHANG JU, AND FUCAI LI

ABSTRACT. We study the incompressible limit of the compressible non-isentropic magnetohydrodynamic equations with zero magnetic diffusivity and general initial data in the whole space  $\mathbb{R}^d$  ( $d = 2, 3$ ). We first establish the existence of classic solutions on a time interval independent of the Mach number. Then, by deriving uniform a priori estimates, we obtain the convergence of the solution to that of the incompressible magnetohydrodynamic equations as the Mach number tends to zero.

## 1. INTRODUCTION

This paper is concerned with the incompressible limit to the compressible non-isentropic magnetohydrodynamic (MHD) equations with zero magnetic diffusivity and general initial data in the whole space  $\mathbb{R}^d$  ( $d = 2, 3$ ).

In the study of a highly conducting fluid, for example, the magnetic fusion, it is rational to ignore the magnetic diffusion term in the MHD equations since the magnetic diffusion coefficient (resistivity coefficient) is inversely proportional to the electrical conductivity coefficient, see [10]. In this situation, the system, describing the motion of the fluid in  $\mathbb{R}^d$ , can be described by the following compressible non-isentropic MHD equations with zero magnetic diffusivity:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \quad (1.1)$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H} + \operatorname{div} \Psi, \quad (1.2)$$

$$\partial_t \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) = 0, \quad \operatorname{div} \mathbf{H} = 0, \quad (1.3)$$

$$\partial_t \mathcal{E} + \operatorname{div}(\mathbf{u}(\mathcal{E}' + p)) = \operatorname{div}((\mathbf{u} \times \mathbf{H}) \times \mathbf{H}) + \operatorname{div}(\mathbf{u} \Psi + \kappa \nabla \theta). \quad (1.4)$$

Here  $\rho$  denotes the density,  $\mathbf{u} \in \mathbb{R}^d$  the velocity,  $\mathbf{H} \in \mathbb{R}^d$  the magnetic field, and  $\theta$  the temperature, respectively;  $\Psi$  is the viscous stress tensor given by

$$\Psi = 2\mu \mathbb{D}(\mathbf{u}) + \lambda \operatorname{div} \mathbf{u} \mathbf{I}_d$$

with  $\mathbb{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)/2$ , and  $\mathbf{I}_d$  being the  $d \times d$  identity matrix, and  $\nabla \mathbf{u}^\top$  the transpose of the matrix  $\nabla \mathbf{u}$ ;  $\mathcal{E}$  is the total energy given by  $\mathcal{E} = \mathcal{E}' + |\mathbf{H}|^2/2$  and  $\mathcal{E}' = \rho(e + |\mathbf{u}|^2/2)$  with  $e$  being the internal energy,  $\rho|\mathbf{u}|^2/2$  the kinetic energy, and  $|\mathbf{H}|^2/2$  the magnetic energy. The viscosity coefficients  $\lambda$  and  $\mu$  of the fluid satisfy  $2\mu + d\lambda > 0$  and  $\mu > 0$ ;  $\kappa > 0$  is the heat conductivity. For simplicity, we assume that  $\mu, \lambda$  and  $\kappa$  are constants. The equations of state  $p = p(\rho, \theta)$  and  $e = e(\rho, \theta)$

---

*Date:* November 15, 2011.

*2000 Mathematics Subject Classification.* 76W05, 35B40.

*Key words and phrases.* Compressible MHD equations, non-isentropic, zero magnetic diffusivity, incompressible limit.

relate the pressure  $p$  and the internal energy  $e$  to the density  $\rho$  and the temperature  $\theta$  of the flow.

For the smooth solution to the system (1.1)–(1.4), we can rewrite the total energy equation (1.4) in the form of the internal energy. In fact, multiplying (1.2) by  $\mathbf{u}$  and (1.3) by  $\mathbf{H}$ , and summing the resulting equations together, we obtain

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{2} |\mathbf{H}|^2 \right) + \frac{1}{2} \operatorname{div}(\rho |\mathbf{u}|^2 \mathbf{u}) + \nabla p \cdot \mathbf{u} \\ = \operatorname{div} \Psi \cdot \mathbf{u} + (\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \mathbf{u} + \nabla \times (\mathbf{u} \times \mathbf{H}) \cdot \mathbf{H}. \end{aligned} \quad (1.5)$$

Using the identities

$$\operatorname{div}(\mathbf{H} \times (\nabla \times \mathbf{H})) = |\nabla \times \mathbf{H}|^2 - \nabla \times (\nabla \times \mathbf{H}) \cdot \mathbf{H}, \quad (1.6)$$

$$\operatorname{div}((\mathbf{u} \times \mathbf{H}) \times \mathbf{H}) = (\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \mathbf{u} + \nabla \times (\mathbf{u} \times \mathbf{H}) \cdot \mathbf{H}, \quad (1.7)$$

and subtracting (1.5) from (1.4), we thus obtain the internal energy equation

$$\partial_t(\rho e) + \operatorname{div}(\rho \mathbf{u} e) + (\operatorname{div} \mathbf{u}) p = \Psi : \nabla \mathbf{u} + \kappa \Delta \theta, \quad (1.8)$$

where  $\Psi : \nabla \mathbf{u}$  denotes the scalar product of two matrices:

$$\Psi : \nabla \mathbf{u} = \sum_{i,j=1}^3 \frac{\mu}{2} \left( \frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \right)^2 + \lambda |\operatorname{div} \mathbf{u}|^2 = 2\mu |\mathbb{D}(\mathbf{u})|^2 + \lambda (\operatorname{tr} \mathbb{D}(\mathbf{u}))^2.$$

Using the Gibbs relation

$$\theta dS = de + p d\left(\frac{1}{\rho}\right), \quad (1.9)$$

we can further replace the equation (1.8) by

$$\partial_t(\rho S) + \operatorname{div}(\rho S \mathbf{u}) = \Psi : \nabla \mathbf{u} + \kappa \Delta \theta, \quad (1.10)$$

where  $S$  denotes the entropy.

In the present paper, we assume that  $\kappa = 0$  in (1.10). Now, as in [27], we reconsider the equations of state as functions of  $S$  and  $p$ , i.e.,  $\rho = R(S, p)$  and  $\theta = \Theta(S, p)$  for some positive smooth functions  $R$  and  $\Theta$  defined for all  $S$  and  $p > 0$ , and satisfying  $\partial R / \partial p > 0$ . For instance, we have  $\rho = p^{1/\gamma} e^{-S/\gamma}$  for ideal fluids. Then, by utilizing (1.1) together with the constraint  $\operatorname{div} \mathbf{H} = 0$ , the system (1.1), (1.2), (1.4) and (1.10) can be rewritten as

$$A(S, p)(\partial_t p + (\mathbf{u} \cdot \nabla) p) + \operatorname{div} \mathbf{u} = 0, \quad (1.11)$$

$$R(S, p)(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H} + \operatorname{div} \Psi, \quad (1.12)$$

$$\partial_t \mathbf{H} - \operatorname{curl}(\mathbf{u} \times \mathbf{H}) = 0, \quad \operatorname{div} \mathbf{H} = 0, \quad (1.13)$$

$$R(S, p) \Theta(S, p) (\partial_t S + (\mathbf{u} \cdot \nabla) S) = \Psi : \nabla \mathbf{u}, \quad (1.14)$$

where

$$A(S, p) = \frac{1}{R(S, p)} \frac{\partial R(S, p)}{\partial p}. \quad (1.15)$$

Considering the physical explanation of the incompressible limit, we introduce the dimensionless parameter  $\epsilon$ , the Mach number, and make the following changes of variables:

$$\begin{aligned} p(x, t) &= p^\epsilon(x, \epsilon t), & S(x, t) &= S^\epsilon(x, \epsilon t), \\ \mathbf{u}(x, t) &= \epsilon \mathbf{u}^\epsilon(x, \epsilon t), & \mathbf{H}(x, t) &= \epsilon \mathbf{H}^\epsilon(x, \epsilon t), \end{aligned}$$

and

$$\mu = \epsilon \mu^\epsilon, \quad \lambda = \epsilon \lambda^\epsilon.$$

As the analysis in [27], we use the transformation  $p^\epsilon(x, \epsilon t) = \underline{p}e^{\epsilon q^\epsilon(x, \epsilon t)}$  for some positive constant  $\underline{p}$ . Under these changes of variables, the system (1.11)–(1.14) becomes

$$a^\epsilon(S^\epsilon, \epsilon q^\epsilon)(\partial_t q^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla)q^\epsilon) + \frac{1}{\epsilon} \operatorname{div} \mathbf{u}^\epsilon = 0, \quad (1.16)$$

$$r^\epsilon(S^\epsilon, \epsilon q^\epsilon)(\partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla)\mathbf{u}^\epsilon) + \frac{1}{\epsilon} \nabla q^\epsilon = (\operatorname{curl} \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon + \operatorname{div} \Psi^\epsilon, \quad (1.17)$$

$$\partial_t \mathbf{H}^\epsilon - \operatorname{curl}(\mathbf{u}^\epsilon \times \mathbf{H}^\epsilon) = 0, \quad \operatorname{div} \mathbf{H}^\epsilon = 0, \quad (1.18)$$

$$b^\epsilon(S^\epsilon, \epsilon q^\epsilon)(\partial_t S^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla)S^\epsilon) = \epsilon^2 \Psi^\epsilon : \nabla \mathbf{u}^\epsilon, \quad (1.19)$$

where we have used the abbreviations  $\Psi^\epsilon = 2\mu^\epsilon \mathbb{D}(\mathbf{u}^\epsilon) + \lambda^\epsilon \operatorname{div} \mathbf{u}^\epsilon \mathbf{I}_d$  and

$$a^\epsilon(S^\epsilon, \epsilon q^\epsilon) := A(S^\epsilon, \underline{p}e^{\epsilon q^\epsilon}) \underline{p}e^{\epsilon q^\epsilon} = \frac{\underline{p}e^{\epsilon q^\epsilon}}{R(S^\epsilon, \underline{p}e^{\epsilon q^\epsilon})} \cdot \frac{\partial R(S^\epsilon, s)}{\partial s} \Big|_{s=\underline{p}e^{\epsilon q^\epsilon}}, \quad (1.20)$$

$$r^\epsilon(S^\epsilon, \epsilon q^\epsilon) := \frac{R(S^\epsilon, \underline{p}e^{\epsilon q^\epsilon})}{\underline{p}e^{\epsilon q^\epsilon}}, \quad b^\epsilon(S^\epsilon, \epsilon q^\epsilon) := R(S^\epsilon, \epsilon q^\epsilon) \Theta(S^\epsilon, \epsilon q^\epsilon). \quad (1.21)$$

Formally, we obtain from (1.16) and (1.17) that  $\nabla q^\epsilon \rightarrow 0$  and  $\operatorname{div} \mathbf{u}^\epsilon = 0$  as  $\epsilon \rightarrow 0$ . Applying the operator *curl* to (1.17), using the fact that  $\operatorname{curl} \nabla = 0$ , and letting  $\epsilon \rightarrow 0$  and  $\mu^\epsilon \rightarrow \mu > 0$ , we obtain

$$\operatorname{curl}(r(\bar{S}, 0)(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - (\operatorname{curl} \bar{\mathbf{H}}) \times \bar{\mathbf{H}} - \mu \Delta \mathbf{v}) = 0,$$

where we have assumed that  $(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)$  and  $r^\epsilon(S^\epsilon, \epsilon q^\epsilon)$  converge to  $(\bar{S}, 0, \mathbf{v}, \bar{\mathbf{H}})$  and  $r(\bar{S}, 0)$  in some sense, respectively. Finally, applying the identity

$$\operatorname{curl}(\mathbf{u} \times \mathbf{H}) = \mathbf{u}(\operatorname{div} \mathbf{H}) - \mathbf{H}(\operatorname{div} \mathbf{u}) + (\mathbf{H} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{H}, \quad (1.22)$$

we expect to get the following incompressible non-isentropic MHD equations

$$r(\bar{S}, 0)(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) - (\operatorname{curl} \bar{\mathbf{H}}) \times \bar{\mathbf{H}} + \nabla \pi = \mu \Delta \mathbf{v}, \quad (1.23)$$

$$\partial_t \bar{\mathbf{H}} + (\mathbf{v} \cdot \nabla) \bar{\mathbf{H}} - (\bar{\mathbf{H}} \cdot \nabla) \mathbf{v} = 0, \quad (1.24)$$

$$\partial_t \bar{S} + (\mathbf{v} \cdot \nabla) \bar{S} = 0, \quad (1.25)$$

$$\operatorname{div} \mathbf{v} = 0, \quad \operatorname{div} \bar{\mathbf{H}} = 0 \quad (1.26)$$

for some function  $\pi$ .

The aim of this paper is to establish the above limit process rigorously in the whole space  $\mathbb{R}^d$ .

Before stating our main results, we review the previous related works. We begin with the results for the Euler and Navier-Stokes equations. For well-prepared initial data, Schochet [33] obtained the convergence of the compressible non-isentropic Euler equations to the incompressible non-isentropic Euler equations in a bounded domain for local smooth solutions. For general initial data, Métivier and Schochet [27] proved rigorously the incompressible limit of the compressible non-isentropic Euler equations in the whole space  $\mathbb{R}^d$ . There are two key points in the article [27]. First, they obtained the uniform estimates in Sobolev norms for the acoustic component of the solutions, which are propagated by a wave equation with unknown variable coefficients. Second, they proved that the local energy of the acoustic wave decays to zero in the whole space case. This approach was extended to the

non-isentropic Euler equations in the exterior domain and the full Navier-Stokes equations in the whole space by Alazard in [1] and [2], respectively, and to the dispersive Navier-Stokes equations by Levermore, Sun and Trivisa [26]. For the spatially periodic case, Métivier and Schochet [28] showed the incompressible limit of the one-dimensional non-isentropic Euler equations with general data. Compared to the non-isentropic case, the treatment of the propagation of oscillations in the isentropic case is simpler and there are many works on this topic. For example, see Ukai [35], Asano [3], Desjardins and Grenier [7] in the whole space; Isozaki [16,17] on the exterior domain; Iguchi [15] on the half space; Schochet [32] and Gallagher [11] in a periodic domain; and Lions and Masmoudi [30], and Desjardins, et al. [8] in a bounded domain. Recently, Jiang and Ou [22] investigated the incompressible limit of the non-isentropic Navier-Stokes equations with zero heat conductivity and well-prepared initial data in three-dimensional bounded domains. The justification of the incompressible limit of the non-isentropic Euler or Navier-Stokes equations with general initial data in a bounded domain or a multi-dimensional periodic domain is still open. The interested reader can refer to [5] for formal computations on the case of viscous polytropic gases and [4, 28] for some analysis on the non-isentropic Euler equations in a multi-dimensional periodic domain. For more results on the incompressible limit of the Euler and Navier-Stokes equations, please see the monograph [9] and the survey articles [6, 31, 34].

For the isentropic compressible MHD equations, the justification of the low Mach limit was given in several aspects. In [23], Klainerman and Majda first studied the incompressible limit of the isentropic compressible ideal MHD equations in the spatially periodic case with well-prepared initial data. Recently, the incompressible limit of the isentropic viscous (including both viscosity and magnetic diffusivity) of compressible MHD equations with general data was studied in [14, 18, 19]. In [14], Hu and Wang obtained the convergence of weak solutions of the compressible viscous MHD equations in bounded, spatially periodic domains and the whole space, respectively. In [18], the authors employed the modulated energy method to verify the limit of weak solutions of the compressible MHD equations in the torus to the strong solution of the incompressible viscous or partial viscous MHD equations (the shear viscosity coefficient is zero but the magnetic diffusion coefficient is a positive constant). In [19], the authors obtained the convergence of weak solutions of the viscous compressible MHD equations to the strong solution of the ideal incompressible MHD equations in the whole space by using the dispersion property of the wave equation if both the shear viscosity and the magnetic diffusion coefficients go to zero.

For the full compressible MHD equations, the incompressible limit in the framework of the so-called variational solutions was studied in [24, 25, 29]. Recently, the authors [20] justified rigorously the low Mach number limit of classical solutions to the ideal or full compressible non-isentropic MHD equations with small entropy or temperature variations. When the heat conductivity and large temperature variations are present, the low Mach number limit for the full compressible non-isentropic MHD equations justified in [21]. We emphasize here that the arguments in [21] are different from the present paper and depend essentially on positivity of fluid viscosity and magnetic diffusivity coefficients.

As aforementioned, in this paper we want to establish rigorously the limit as  $\epsilon \rightarrow 0$  to the system (1.16)–(1.19) for  $\mu_\epsilon \rightarrow \mu > 0$ . In this case, the magnetic

equation is purely hyperbolic due to the lack of magnetic diffusivity. The first-order derivatives of  $\mathbf{H}^\epsilon$  in the momentum equation and magnetic equation cannot be controlled. It is very hard to study the system (1.16)–(1.19). To the author's knowledge, there is very few mathematical analysis on the system (1.16)–(1.19) with fixed or unfixed  $\epsilon$ , even for the isentropic case. Our main idea is trying to make full use of the fluid viscosities to control the higher order derivatives of the magnetic field.

Now, we supplement the system (1.16)–(1.19) with initial conditions

$$(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)|_{t=0} = (S_0^\epsilon, q_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{H}_0^\epsilon) \quad (1.27)$$

and state the main results as follows.

**Theorem 1.1.** *Let  $s > d/2 + 2$  be an integer. Assume that  $\mu^\epsilon \rightarrow \mu > 0$  and  $\lambda^\epsilon \rightarrow \lambda$  as  $\epsilon \rightarrow 0$ . Suppose that the initial data  $(S_0^\epsilon, q_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{H}_0^\epsilon)$  satisfy*

$$\|(S_0^\epsilon, q_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{H}_0^\epsilon)\|_{H^s(\mathbb{R}^d)} \leq M_0. \quad (1.28)$$

*Then there exists a  $T > 0$  such that for any  $\epsilon \in (0, 1]$ , the Cauchy problem (1.16)–(1.19), (1.27) has a unique solution  $(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon) \in C^0([0, T], H^s(\mathbb{R}^d))$ , and there exists a positive constant  $N$ , depending only on  $T$  and  $M_0$ , such that*

$$\|(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)(t)\|_{H^s(\mathbb{R}^d)} \leq N, \quad \forall t \in [0, T]. \quad (1.29)$$

*Furthermore, if there exist positive constants  $\underline{S}$ ,  $N_0$  and  $\delta$  such that  $S_0^\epsilon(x)$  satisfies*

$$|S_0^\epsilon(x) - \underline{S}| \leq N_0|x|^{-1-\delta}, \quad |\nabla S_0^\epsilon(x)| \leq N_0|x|^{-2-\delta}, \quad (1.30)$$

*then the sequence of solutions  $(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)$  converges weakly in  $L^\infty(0, T; H^s(\mathbb{R}^d))$  and strongly in  $L^2(0, T; H_{\text{loc}}^{s'}(\mathbb{R}^d))$  for all  $s' < s$  to a limit  $(\bar{S}, 0, \mathbf{v}, \bar{\mathbf{H}})$ , where  $(\bar{S}, \mathbf{v}, \bar{\mathbf{H}})$  is the unique solution in  $C([0, T], H^s(\mathbb{R}^d))$  of (1.23)–(1.26) with initial data  $(\bar{S}, \mathbf{v}, \bar{\mathbf{H}})|_{t=0} = (S_0, \mathbf{w}_0, \mathbf{H}_0)$ , where  $\mathbf{w}_0 \in H^s(\mathbb{R}^d)$  is determined by*

$$\operatorname{div} \mathbf{w}_0 = 0, \quad \operatorname{curl}(r(S_0, 0)\mathbf{w}_0) = \operatorname{curl}(r(S_0, 0)\mathbf{v}_0), \quad r(S_0, 0) := \lim_{\epsilon \rightarrow 0} r^\epsilon(S_0^\epsilon, 0). \quad (1.31)$$

*The function  $\pi \in C([0, T] \times \mathbb{R}^d)$  satisfies  $\nabla \pi \in C([0, T], H^{s-1}(\mathbb{R}^d))$ .*

We briefly describe the strategy of the proof. The proof of Theorems 1.1 includes two main steps: the uniform estimates of the solutions, and the convergence from the original scaling equations to the limiting ones. Once we have established the uniform estimates (1.29) of the solutions in Theorems 1.1, the convergence of solutions is easily proved by using the local energy decay theorem for fast waves in the whole space, which is shown by Métivier and Schochet in [27]. Thus, the main task in the present paper is to obtain the uniform estimates (1.29). For this purpose, we shall modify the approach developed in [27]. In fact, due to the strong coupling of hydrodynamic motion and magnetic field, and the lack of magnetic diffusivity, new difficulties arise in obtaining the uniform estimates for the solutions to (1.16)–(1.19), (1.27). First of all, when we perform the operator  $(\{E^\epsilon\}^{-1}L(\partial_x))^\sigma$  to the continuity and momentum equations, or the operator  $\operatorname{curl}$  to the momentum equations, one order more spatial derivatives arise for the magnetic field, and this prevents us from closing the energy estimates. Second, since the coefficients of the acoustic wave equations depend on the entropy, we could not get the estimates of  $\|\mathbf{u}^\epsilon\|_{L^2(0, T; H^{s+1})}$  directly from the system. The ideas to overcome these difficulties here are the following: We transfer one spatial derivative from the magnetic field to the velocity with the help of the special coupled way between magnetic field and

fluid velocity. Then, to control  $\|\mathbf{u}^\epsilon\|_{L^2(0,T;H^{s+1})}$ , we employ a kind of Helmholtz decomposition of the velocity. Third, we make full use the special structure of the magnetic field equation and the estimates on  $\mathbf{u}$  to control  $\|\mathbf{H}^\epsilon\|_{L^\infty(0,T;H^s)}$ .

We point out that our arguments in this paper can be modified slightly to the case of the the compressible non-isentropic MHD equations with infinite Reynolds number. We shall give a brief discussion in Section 5.

This paper is arranged as follows. In Section 2, we give notations, recall basic facts, and present commutators estimates. In Section 3 we establish the uniform boundness of the solutions and prove the existence part of Theorem 1.1. In Section 4, we use the decay of the local energy to the acoustic wave equations to prove the convergent part of Theorem 1.1. In the last section, we consider the incompressible limit to the compressible non-isentropic MHD equations with infinite Reynolds number.

## 2. PRELIMINARY

We give notations and recall basic facts which will be used frequently in the proofs.

(1) We denote  $\langle \cdot, \cdot \rangle$  the standard inner product in  $L^2(\mathbb{R}^d)$  with  $\langle f, f \rangle = \|f\|^2$  and  $H^k$  the usual Sobolev space  $W^{k,2}$  with norm  $\|\cdot\|_k$ , in particular,  $\|\cdot\|_0 = \|\cdot\|$ . The notation  $\|(A_1, \dots, A_k)\|$  means the summation of  $\|A_i\|$  ( $i = 1, \dots, k$ ), and it also applies to other norms. For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_d)$ , we denote  $D^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$  and  $|\alpha| = |\alpha_1| + \dots + |\alpha_d|$ . We also omit the spatial domain  $\mathbb{R}^d$  in integrals for convenience. We use the symbols  $K$  or  $C_0$  to denote the generic positive constants, and  $C(\cdot)$  and  $\tilde{C}(\cdot)$  to denote the smooth functions, which may vary from line to line.

(2) For a scalar function  $f$  and vector functions  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$ , we have the following basic vector identities:

$$\operatorname{div}(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \operatorname{curl} \mathbf{a} - \mathbf{a} \cdot \operatorname{curl} \mathbf{b}, \quad (2.1)$$

$$\nabla(|\mathbf{a}|^2) = 2(\mathbf{a} \cdot \nabla) \mathbf{a} + 2\mathbf{a} \times \operatorname{curl} \mathbf{a}, \quad (2.2)$$

$$\operatorname{curl}(f\mathbf{a}) = f \cdot \operatorname{curl} \mathbf{a} - \nabla f \times \mathbf{a}, \quad (2.3)$$

$$\operatorname{curl}(\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b} + \mathbf{a}(\operatorname{div} \mathbf{b}) - \mathbf{b}(\operatorname{div} \mathbf{a}), \quad (2.4)$$

$$\operatorname{div}((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}) = \mathbf{c} \cdot \operatorname{curl}(\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times \mathbf{b}) \cdot \operatorname{curl} \mathbf{c}. \quad (2.5)$$

(3) We have the following well-known nonlinear estimates [12].

(i) Let  $\alpha = (\alpha_1, \alpha_2, \alpha_d)$  be a multi-index such that  $|\alpha| = k$ . Then, for all  $\sigma \geq 0$ , and  $f, g \in H^{k+\sigma}(\mathbb{R}^d)$ , there exists a generic constant  $C_0$  such that

$$\|[f, \partial^\alpha]g\|_{H^\sigma} \leq C_0(\|f\|_{W^{1,\infty}} \|g\|_{H^{\sigma+k-1}} + \|f\|_{H^{\sigma+k}} \|g\|_{L^\infty}). \quad (2.6)$$

(ii) For integers  $k \geq 0$ ,  $l \geq 0$ ,  $k+l \leq \sigma$  and  $\sigma > d/2$ , the product maps continuously  $H^{\sigma-k}(\mathbb{R}^d) \times H^{\sigma-l}(\mathbb{R}^d)$  to  $H^{\sigma-k-l}(\mathbb{R}^d)$  and

$$\|uv\|_{\sigma-k-l} \leq K\|u\|_{\sigma-k}\|v\|_{\sigma-l}. \quad (2.7)$$

(iii) Let  $\sigma > d/2$  be an integer. Assume that  $F(u)$  is a smooth function such that  $F(0) = 0$  and  $u \in H^\sigma(\mathbb{R}^d)$ , then  $F(u) \in H^\sigma(\mathbb{R}^d)$  and its norm is bounded by

$$\|F(u)\|_\sigma \leq C(\|u\|_\sigma)\|u\|_\sigma, \quad (2.8)$$

where  $C(\cdot)$  is independent of  $u$  and maps  $[0, \infty)$  into  $[0, \infty)$ .

## 3. UNIFORM ESTIMATES

In this section and the first part of the next section we assume that  $\mu^\epsilon \equiv \mu > 0$  and  $\lambda^\epsilon \equiv \lambda$  for simplicity of the presentation. The general case can be treated by a slight modification in the arguments presented here.

In view of [27] and the classical local existence results obtained by Vol'pert and Khudiaev [36] for hyperbolic-parabolic systems, the key point in the proof of the existence part of Theorem 1.1 is to establish the uniform estimate (1.29), which can be deduced from the following *a priori* estimate.

**Theorem 3.1.** *For any  $\epsilon > 0$ , let  $(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon) \in C([0, T], H^s(\mathbb{R}^d))$  be the solution to (1.16)–(1.19). Then there exists an increasing  $C(\cdot)$  from  $[0, \infty)$  to  $[0, \infty)$ , such that*

$$\mathcal{M}_\epsilon(T) \leq C_0 + (T + \epsilon)C(\mathcal{M}_\epsilon(T)), \quad (3.1)$$

where

$$\mathcal{M}_\epsilon(T) := \mathcal{N}_\epsilon(T)^2 + \int_0^T \|\mathbf{u}^\epsilon\|_{s+1}^2 d\tau, \quad (3.2)$$

with

$$\mathcal{N}_\epsilon(T) := \sup_{t \in [0, T]} \|(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)(t)\|_s. \quad (3.3)$$

The remainder of this section is devoted to establishing (3.1). In the calculations that follow, we always suppose that the assumptions in Theorem 1.1 hold. We consider a solution  $(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)$  to the problem (1.16)–(1.19), (1.27) on  $C([0, T], H^s(\mathbb{R}^d))$  with initial data satisfying (1.28).

The main idea for proving the uniform estimate (3.1) is motivated by the work [27] where the operator  $(\{E^\epsilon\}^{-1}L(\partial_x))^m$  is introduced to control the acoustic components of velocity for the Euler equations. When the strong coupling of the fluid and magnetic field is present, however, the arguments in [27] cannot be directly applied to get a uniform estimate of the acoustic parts due to lack of magnetic diffusion in the magnetic equation. Instead, here we transfer one order spatial derivative from  $\mathbf{H}^\epsilon$  to  $\mathbf{u}^\epsilon$ , and then employ the fluid viscosity to control higher derivatives. We remark that the reason that these techniques work is due to the special structure of coupling between the fluid and magnetic fields.

We begin with the estimate on the entropy  $S^\epsilon$ .

**Lemma 3.2.** *There exist a constant  $C_0 > 0$  and a function  $C(\cdot)$ , independent of  $\epsilon$ , such that for all  $t \in (0, T]$ ,*

$$\|S^\epsilon(t)\|_s^2 \leq C_0 + tC(\mathcal{M}_\epsilon(T)) + \epsilon^2 C(\mathcal{M}_\epsilon(T)). \quad (3.4)$$

*Proof.* For the multi-index  $\alpha$  satisfying  $|\alpha| \leq s-1$ , denote  $f_\alpha = \partial_x^\alpha S^\epsilon$ . In view of the positivity of  $b^\epsilon(S^\epsilon, \epsilon q^\epsilon)$ , we deduce from (1.19) that,

$$\partial_t f_\alpha + (\mathbf{u}^\epsilon \cdot \nabla) f_\alpha = g_\alpha + \epsilon^2 h_\alpha, \quad (3.5)$$

where

$$g_\alpha = -[\partial_x^\alpha, \mathbf{u}^\epsilon] \cdot \nabla S^\epsilon, \quad h_\alpha = \epsilon^2 \partial_x^\alpha \left( \frac{\Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon q^\epsilon)} \right).$$

The commutator inequality (2.6) and Sobolev embedding theorem imply that  $\|g_\alpha\| \leq C(M_\epsilon(T))$ . On the other hand, from the Sobolev embedding theorem and the Moser-type inequality [23] we get

$$\begin{aligned} \|h_\alpha\| &\leq K \left( \|(\Psi^\epsilon : \nabla \mathbf{u}^\epsilon)\|_{L^\infty} \left\| D^s \left( \frac{1}{b^\epsilon} \right) \right\| + \|D^s(\Psi^\epsilon : \nabla \mathbf{u}^\epsilon)\| \left\| \frac{1}{b^\epsilon} \right\|_{L^\infty} \right) \\ &\leq C(\mathcal{N}_\epsilon(T)) + C(\mathcal{N}_\epsilon(T)) \|\mathbf{u}^\epsilon\|_{s+1}. \end{aligned}$$

Multiplying (3.5) by  $f_\alpha$  and integrating over  $[0, t] \times \mathbb{R}^d$  with  $t \leq T$ , we obtain

$$\begin{aligned} \|f_\alpha(t)\|^2 &\leq \|f_\alpha(0)\|^2 + \|\partial_x \mathbf{u}^\epsilon\|_{L^\infty((0,t) \times \mathbb{R}^d)} \int_0^t \|f_\alpha(\tau)\|^2 d\tau \\ &\quad + 2 \int_0^t \|g_\alpha(\tau)\| \|f_\alpha(\tau)\| d\tau + 2\epsilon^2 \int_0^t \|h_\alpha(\tau)\| \|f_\alpha(\tau)\| d\tau \\ &\leq C_0 + tC(\mathcal{M}_\epsilon(T)) + \epsilon^2 C(\mathcal{M}_\epsilon(T)), \end{aligned}$$

where we have used Young's inequality and the embedding  $H^\sigma \hookrightarrow L^\infty$  for  $\sigma > d/2$ . The conclusion then follows by adding up these estimates for all  $|\alpha| \leq s-1$ .  $\square$

The following  $L^2$ -bound of  $(q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)$  can be obtained directly using the energy method due to the skew-symmetry of the singular term in the system and the special structure of coupling between the magnetic field and fluid velocity. This  $L^2$ -bound is very important in our arguments, since the induction analysis will be used to get the desired Sobolev estimates.

**Lemma 3.3.** *There exist constants  $C_0 > 0$  and  $0 < \xi_1 < \mu$ , and a function  $C(\cdot)$  independent of  $\epsilon$ , such that for all  $t \in [0, T]$ ,*

$$\|(q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)(t)\|^2 + \xi_1 \int_0^t \|\nabla \mathbf{u}^\epsilon(\tau)\|^2 d\tau \leq C_0 + tC(\mathcal{M}_\epsilon(T)). \quad (3.6)$$

*Proof.* Multiplying (1.16) by  $q^\epsilon$ , (1.17) by  $\mathbf{u}^\epsilon$ , and (1.18) by  $\mathbf{H}^\epsilon$ , respectively, integrating over  $\mathbb{R}^d$ , and adding the resulting equations together, we obtain

$$\begin{aligned} &\langle a^\epsilon \partial_t q^\epsilon, q^\epsilon \rangle + \langle r^\epsilon \partial_t \mathbf{u}^\epsilon, \mathbf{u}^\epsilon \rangle + \langle \partial_t \mathbf{H}^\epsilon, \mathbf{H}^\epsilon \rangle + \mu \|\nabla \mathbf{u}^\epsilon\|^2 + (\mu + \lambda) \|\operatorname{div} \mathbf{u}^\epsilon\|^2 \\ &\quad + \langle a^\epsilon (\mathbf{u}^\epsilon \cdot \nabla) q^\epsilon, q^\epsilon \rangle + \langle r^\epsilon (\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon, \mathbf{u}^\epsilon \rangle \\ &= \int [(\operatorname{curl} \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon] \cdot \mathbf{u}^\epsilon dx + \int \operatorname{curl}(\mathbf{u}^\epsilon \times \mathbf{H}^\epsilon) \cdot \mathbf{H}^\epsilon dx. \end{aligned} \quad (3.7)$$

Here the singular terms involving  $1/\epsilon$  are canceled. Using the identity (1.7) and integrating by parts, we immediately obtain that

$$\int [(\operatorname{curl} \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon] \cdot \mathbf{u}^\epsilon dx + \int \operatorname{curl}(\mathbf{u}^\epsilon \times \mathbf{H}^\epsilon) \cdot \mathbf{H}^\epsilon dx = 0.$$

In view of the positivity and smoothness of  $a^\epsilon(S^\epsilon, \epsilon q^\epsilon)$  and  $r^\epsilon(S^\epsilon, \epsilon q^\epsilon)$ , we get directly from (1.16), (1.19) and (2.8) that

$$\|\partial_t S^\epsilon\|_{s-1} \leq C(\mathcal{N}_\epsilon(T)), \quad \|\epsilon \partial_t q^\epsilon\|_{s-1} \leq C(\mathcal{N}_\epsilon(T)), \quad (3.8)$$

while by the Sobolev embedding theorem, we find that

$$\|(\partial_t a^\epsilon, \partial_t r^\epsilon)\|_{L^\infty} \leq \|(\partial_t a^\epsilon, \partial_t r^\epsilon)\|_{s-2} \leq C(\mathcal{N}_\epsilon(T)). \quad (3.9)$$

By the definition of  $\mathcal{N}_\epsilon(T)$  and the Sobolev embedding theorem, it is easy to see that

$$\|(\nabla a^\epsilon, \nabla r^\epsilon)\|_{L^\infty} \leq C(\mathcal{N}_\epsilon(T)).$$



Since  $\mu > 0, 2\mu + d\lambda > 0$ , there exists a positive constant  $\kappa_1$  such that

$$\mu \|\nabla \mathbf{u}^\epsilon\|^2 + (\mu + \lambda) \|\operatorname{div} \mathbf{u}^\epsilon\|^2 \geq \kappa_1 \|\nabla \mathbf{u}^\epsilon\|^2.$$

Thus, from (3.7) we get that

$$\begin{aligned} & \langle a^\epsilon q^\epsilon, q^\epsilon \rangle + \langle r^\epsilon \mathbf{u}^\epsilon, \mathbf{u}^\epsilon \rangle + \langle \mathbf{H}^\epsilon, \mathbf{H}^\epsilon \rangle + \kappa_1 \int_0^t \|\nabla \mathbf{u}^\epsilon(\tau)\|^2 d\tau \\ & \leq \left\{ \langle a^\epsilon q^\epsilon, q^\epsilon \rangle + \langle r^\epsilon \mathbf{u}^\epsilon, \mathbf{u}^\epsilon \rangle + \langle \mathbf{H}^\epsilon, \mathbf{H}^\epsilon \rangle \right\} \Big|_{t=0} \\ & \quad + C(\mathcal{M}_\epsilon(T)) \int_0^t \{|q^\epsilon(\tau)|^2 + |\mathbf{u}^\epsilon(\tau)|^2 + |\mathbf{H}^\epsilon(\tau)|^2\} d\tau. \end{aligned} \quad (3.10)$$

Moreover, we have

$$\begin{aligned} \|q^\epsilon\|^2 + \|\mathbf{u}^\epsilon\|^2 & \leq \|(a^\epsilon)^{-1}\|_{L^\infty} \langle a^\epsilon q^\epsilon, q^\epsilon \rangle + \|(r^\epsilon)^{-1}\|_{L^\infty} \langle r^\epsilon \mathbf{u}^\epsilon, \mathbf{u}^\epsilon \rangle \\ & \leq C_0 (\langle a^\epsilon q^\epsilon, q^\epsilon \rangle + \langle r^\epsilon \mathbf{u}^\epsilon, \mathbf{u}^\epsilon \rangle), \end{aligned} \quad (3.11)$$

since  $a^\epsilon$  and  $r^\epsilon$  are uniformly bounded away from zero. Applying Gronwall's Lemma to (3.10), we conclude

$$\|(q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)(t)\|^2 \leq C_0 \|(q_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{H}_0^\epsilon)\|^2 \exp\{tC(\mathcal{M}_\epsilon(T))\}.$$

Therefore, the estimate (3.6) follows from an elementary inequality

$$e^{Ct} \leq 1 + \tilde{C}t, \quad 0 \leq t \leq T_0, \quad (3.12)$$

where  $T_0$  is some fixed constant.  $\square$

Concerning the desired higher order estimates, we cannot directly get them by differentiating the system as done in [27], since the coefficients  $a^\epsilon(S^\epsilon, \epsilon q^\epsilon)$ ,  $r^\epsilon(S^\epsilon, \epsilon q^\epsilon)$  and  $b^\epsilon(S^\epsilon, \epsilon q^\epsilon)$  contain two scales  $S^\epsilon$  and  $\epsilon q^\epsilon$ . We shall adapt and modify the techniques developed in [27] to derive the higher order estimates. Set

$$\begin{aligned} E^\epsilon(S^\epsilon, \epsilon q^\epsilon) &= \begin{pmatrix} a^\epsilon(S^\epsilon, \epsilon q^\epsilon) & 0 \\ 0 & r^\epsilon(S^\epsilon, \epsilon q^\epsilon) \mathbf{I}_d \end{pmatrix}, \quad \mathbf{U}^\epsilon = \begin{pmatrix} q^\epsilon \\ \mathbf{u}^\epsilon \end{pmatrix}, \\ L(\partial_x) &= \begin{pmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{pmatrix}, \end{aligned}$$

where  $\mathbf{I}_d$  denotes the  $d \times d$  unit matrix.

Let  $L_{E^\epsilon}(\partial_x) = \{E^\epsilon\}^{-1} L(\partial_x)$  and  $r_0(S^\epsilon) = r^\epsilon(S^\epsilon, 0)$ . Note that  $r_0(S^\epsilon)$  is smooth, positive, and bounded away from zero with respect to each  $\epsilon$ . First, using Lemma 3.2 and employing the same analysis as in [27], we have

**Lemma 3.4.** *There exist constants  $C_1 > 0$ ,  $K > 0$ , and a function  $C(\cdot)$ , depending only on  $M_0$ , such that for all  $\sigma \in [1, \dots, s]$  and  $t \in [0, T]$ ,*

$$\|\mathbf{U}^\epsilon\|_\sigma \leq K \|L(\partial_x) \mathbf{U}^\epsilon\|_{\sigma-1} + \tilde{C} (\|\operatorname{curl}(r_0 \mathbf{u}^\epsilon)\|_{\sigma-1} + \|\mathbf{U}^\epsilon\|_{\sigma-1}) \quad (3.13)$$

and

$$\|\mathbf{U}^\epsilon\|_\sigma \leq \tilde{C} \{ \|\{L_{E^\epsilon}(\partial_x)\}^\sigma \mathbf{U}^\epsilon\|_0 + \|\operatorname{curl}(r_0 \mathbf{u}^\epsilon)\|_{\sigma-1} + \|\mathbf{U}^\epsilon\|_{\sigma-1} \}, \quad (3.14)$$

where  $\tilde{C} := C_1 + tC(\mathcal{M}_\epsilon(T)) + \epsilon C(\mathcal{M}_\epsilon(T))$ .

We remark that the inequalities (3.13) and (3.14) are similar to the well known Helmholtz decomposition, and the estimate on  $\|S^\epsilon(t)\|_s^2$  in Lemma 3.2 plays a key role in the proof of Lemma 3.4.

Our next task is to derive a bound on  $\|\{L_{E^\epsilon}(\partial_x)\}^\sigma \mathbf{U}^\epsilon\|_0$  and  $\|\operatorname{curl}(r_0 \mathbf{u}^\epsilon)\|_{\sigma-1}$  by induction arguments. Let  $\mathbf{W}_\sigma^\epsilon = \{L_{E^\epsilon}(\partial_x)\}^\sigma(0, \mathbf{u}^\epsilon)^\top$ . We first show the following estimate.

**Lemma 3.5.** *There exist a sufficiently small constant  $\eta_1 > 0$  and two constants  $C_0 > 0$ ,  $0 < \xi_2 < \mu$ , and a function  $C(\cdot)$  from  $[0, \infty)$  to  $[0, \infty)$ , independent of  $\epsilon$ , such that for all  $\sigma \in [1, \dots, s]$  and  $t \in [0, T]$ ,*

$$\begin{aligned} & \|\{L_{E^\epsilon}(\partial_x)\}^\sigma \mathbf{U}^\epsilon(t)\|^2 + \frac{\xi_2}{2} \int_0^t \|\nabla \mathbf{W}_\sigma^\epsilon\|^2(\tau) d\tau \\ & \leq C_0 + tC(\mathcal{N}_\epsilon(T)) + \eta_1 \int_0^t \|\mathbf{u}^\epsilon(\tau)\|_{s+1}^2 d\tau. \end{aligned} \quad (3.15)$$

*Proof.* Let  $\mathbf{U}_\sigma^\epsilon := \{L_{E^\epsilon}(\partial_x)\}^\sigma \mathbf{U}^\epsilon$ ,  $\sigma \in \{0, \dots, s\}$ . For simplicity, we set  $\mathcal{M} := \mathcal{M}_\epsilon(T)$ ,  $\mathcal{N} := \mathcal{N}_\epsilon(T)$ , and  $E := E^\epsilon(S^\epsilon, \epsilon q^\epsilon)$ . The case  $k = 0$  is an immediate consequence of Lemma 3.3. It is easy to verify that the operator  $L_E(\partial_x)$  is bounded from  $H^k$  to  $H^{k-1}$  for  $k \in \{1, \dots, s+1\}$ . Note that the equations (1.16), (1.17) can be written as

$$(\partial_t + \mathbf{u}^\epsilon \cdot \nabla) \mathbf{U}^\epsilon + \frac{1}{\epsilon} E^{-1} L(\partial_x) \mathbf{U}^\epsilon = E^{-1} (\mathbf{J}^\epsilon + \mathbf{V}^\epsilon) \quad (3.16)$$

with

$$\mathbf{J}^\epsilon = \begin{pmatrix} 0 \\ (\operatorname{curl} \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon \end{pmatrix}, \quad \mathbf{V}^\epsilon = \begin{pmatrix} 0 \\ \operatorname{div} \Psi^\epsilon \end{pmatrix}.$$

For  $\sigma \geq 1$ , we commute the operator  $\{L_E\}^\sigma$  with (3.16) and multiply the resulting equation by  $E$  to infer that

$$E(\partial_t + \mathbf{u}^\epsilon \cdot \nabla) \mathbf{U}_\sigma^\epsilon + \frac{1}{\epsilon} L(\partial_x) \mathbf{U}_\sigma^\epsilon = E(\mathbf{f}_\sigma + \mathbf{g}_\sigma + \mathbf{h}_\sigma), \quad (3.17)$$

where

$$\begin{aligned} \mathbf{f}_\sigma &:= [\partial_t + \mathbf{u}^\epsilon \cdot \nabla, \{L_E\}^\sigma] \mathbf{U}^\epsilon, \\ \mathbf{g}_\sigma &:= \{L_E\}^\sigma (E^{-1} \mathbf{J}^\epsilon), \\ \mathbf{h}_\sigma &:= \{L_E\}^\sigma (E^{-1} \mathbf{V}^\epsilon). \end{aligned}$$

Multiplying (3.17) by  $\mathbf{U}_\sigma^\epsilon$  and integrating over  $[0, t] \times \mathbb{R}^d$  with  $t \leq T$ , noticing that the singular terms cancel out since  $L(\partial_x)$  is skew-adjoint, we use the inequalities (3.8) and (3.9), and Cauchy-Schwarz's inequality to deduce that

$$\begin{aligned} \langle E(t) \mathbf{U}_\sigma^\epsilon(t), \mathbf{U}_\sigma^\epsilon(t) \rangle & \leq \langle E(0) \mathbf{U}_\sigma^\epsilon(0), \mathbf{U}_\sigma^\epsilon(0) \rangle + C(\mathcal{M}) \int_0^t \|\mathbf{U}_\sigma^\epsilon(\tau)\|^2 d\tau \\ & \quad + \int_0^t \|\mathbf{f}_\sigma(\tau)\|^2 d\tau + 2 \int_0^t \int_{\mathbb{R}^d} (E(\mathbf{g}_\sigma + \mathbf{h}_\sigma) \mathbf{U}_\sigma^\epsilon)(\tau) d\tau. \end{aligned} \quad (3.18)$$

Following the proof process of Lemma 2.4 in [27], we obtain that

$$\|\mathbf{f}_k(t)\| \leq C(\mathcal{N}_\epsilon(t)). \quad (3.19)$$

Now we estimate the nonlinear term in (3.18) involving  $\mathbf{g}_\sigma$ . We expand  $\mathbf{g}_\sigma$  as follows

$$\mathbf{g}_\sigma = \sum_{i,j=1}^d \sum_{|\alpha|=\sigma+1} \{E^{-1}\}^{k+1} \partial_x^\alpha H_i^\epsilon H_j^\epsilon$$

$$\begin{aligned}
 & + \sum_{i,j=1}^d \sum_{\Lambda_1} \sum_{\Lambda_2} \{E^{-1}\}^l \partial_x^{\beta_1} \{E^{-1}\} \cdots \partial_x^{\beta_k} \{E^{-1}\} \partial_x^\gamma H_i^\epsilon \partial_x^\delta H_j^\epsilon \\
 & := B_1 + B_2,
 \end{aligned}$$

where

$$\begin{aligned}
 \Lambda_1 &= \{(\beta_1, \dots, \beta_k, \gamma, \delta) \mid |\beta_1| + \dots + |\beta_k| + |\gamma| + |\delta| \leq k+1, 0 < |\gamma| \leq k, |\delta| \leq k\}, \\
 \Lambda_2 &= \{l \mid l = k+1 - (|\beta_1| + \dots + |\beta_k|), (\beta_1, \dots, \beta_k, 0, 0) \in \Lambda_1\}.
 \end{aligned}$$

Since there is no magnetic diffusion in the system, we cannot deal with directly the terms involving  $B_1$ . Instead, we transform one spatial derivative to  $\mathbf{U}_\sigma^\epsilon$ . Integrating by parts, we have

$$\begin{aligned}
 \int_{\mathbb{R}^d} (EB_1 \mathbf{U}_\sigma^\epsilon)(\tau) dx &= - \sum_{i,j=1}^d \sum_{|\alpha|=\sigma} \int_{\mathbb{R}^d} \{E^{-1}\}^k \partial_x^\alpha H_i^\epsilon \partial_x H_j^\epsilon \mathbf{U}_\sigma^\epsilon dx \\
 &\quad - \sum_{i,j=1}^d \sum_{|\alpha|=\sigma} \int_{\mathbb{R}^d} \partial_x \{E^{-1}\}^k \partial_x^\alpha H_i^\epsilon H_j^\epsilon \mathbf{U}_\sigma^\epsilon dx \\
 &\quad - \sum_{i,j=1}^d \sum_{|\alpha|=\sigma} \int_{\mathbb{R}^d} \{E^{-1}\}^k \partial_x^\alpha H_i^\epsilon H_j^\epsilon \partial_x (\mathbf{U}_\sigma^\epsilon) dx \\
 &\leq C(\mathcal{N}) + \eta_1 \|\mathbf{u}^\epsilon(\tau)\|_{s+1}^2
 \end{aligned} \tag{3.20}$$

for sufficiently small constant  $\eta_1 > 0$ .

By virtue of Cauchy-Schwarz's and Sobolev's inequalities, and (3.9), a direct computation implies that

$$\int_{\mathbb{R}^d} |(EB_2 \mathbf{U}_\sigma^\epsilon)(\tau)| d\tau \leq C(\mathcal{N}). \tag{3.21}$$

Next, we deal with the term involving the viscosity. Recall that  $L(\partial_x) \mathbf{U}^\epsilon = (\operatorname{div} \mathbf{u}^\epsilon, \nabla q^\epsilon)$ . Denote

$$L_1 := \{a^\epsilon\}^{-1} \operatorname{div}, \quad L_2 := \{r^\epsilon\}^{-1} \nabla.$$

A straightforward computation implies that

$$\mathbf{U}_k^\epsilon = \begin{cases} \begin{pmatrix} \{L_1 L_2\}^{\frac{k-1}{2}} L_1 \mathbf{u}^\epsilon \\ \{L_2 L_1\}^{\frac{k-1}{2}} L_2 q^\epsilon \end{pmatrix}, & \text{if } k \text{ is odd;} \\ \begin{pmatrix} \{L_1 L_2\}^{k/2} q^\epsilon \\ \{L_2 L_1\}^{k/2} \mathbf{u}^\epsilon \end{pmatrix}, & \text{if } k \text{ is even.} \end{cases}$$

Thus, we induce that

$$\int_0^t \int_{\mathbb{R}^d} (E \mathbf{h}_\sigma \mathbf{U}_\sigma^\epsilon)(\tau) dx d\tau = \int_0^t \int_{\mathbb{R}^d} E \{L_E\}^\sigma (E^{-1} \mathbf{V}^\epsilon) \mathbf{W}_\sigma^\epsilon dx d\tau.$$

An integration by parts gives

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{R}^d} E L_E^\sigma (E^{-1} \mathbf{V}^\epsilon) \mathbf{W}_\sigma^\epsilon dx d\tau \\
 &= - \int_0^t \int_{\mathbb{R}^d} \mu |\nabla \mathbf{W}_\sigma^\epsilon|^2 + (\mu + \lambda) |\operatorname{div} \mathbf{W}_\sigma^\epsilon|^2 dx d\tau
 \end{aligned}$$

$$+ \int_0^t \int_{\mathbb{R}^d} E[\mu E^{-1} \Delta + (\mu + \lambda) E^{-1} \nabla \operatorname{div}, \{L_E\}^\sigma](0, \mathbf{u}^\epsilon)^T \mathbf{W}_\sigma^\epsilon dx d\tau,$$

where it is easy to verify that

$$[E^{-1} \Delta, \{L_E\}^\sigma] = \sum_{i=0}^{k-1} \{L_E\}^i [E^{-1} \Delta, L_E] \{L_E\}^{\sigma-i-1}. \quad (3.22)$$

Noting that  $L_E(\partial_x) = E^{-1} L(\partial_x)$ , we find that

$$[E^{-1} \Delta, L_E] = -E^{-1} \Delta E^{-1} L(\partial_x) + \sum_{i,j=1}^d B_{ij} \partial_{x_{ij}},$$

where  $B_{ij}$  ( $i, j = 1, \dots, d$ ) are the sums of bilinear functions of  $E^{-1}$  and  $\partial_x \{E^{-1}\}$ , and Sobolev's inequalities imply that

$$\|B_{ij}\|_{s-1} \leq C(\mathcal{N}).$$

Thus, we integrate by parts to infer that

$$\begin{aligned} & \mu \int_0^t \int_{\mathbb{R}^d} E[E^{-1} \Delta, \{L_E\}^\sigma](0, \mathbf{u}^\epsilon)^\top \mathbf{W}_\sigma^\epsilon dx d\tau \\ & \leq \frac{\mu}{2} \int_0^t \|\nabla \mathbf{W}_\sigma^\epsilon\|^2 d\tau + C(\mathcal{N}) \int_0^t \|\mathbf{W}_\sigma^\epsilon(\tau)\|^2 d\tau \\ & \quad + \int_0^t \|\tilde{H}_2^\sigma(\tau)\|^2 + \|\tilde{H}_1^\sigma(\tau)\|^2 d\tau. \end{aligned}$$

Here  $\tilde{H}_1^\sigma$  is a finite sum of terms of the form

$$(\partial_x^{\alpha_1} e_1) \cdots (\partial_x^{\alpha_l} e_l) (\partial_x^\beta w) (\partial_x^\gamma u_m^\epsilon)$$

with  $|\alpha_1| + \cdots + |\alpha_l| + |\beta| + |\gamma| \leq \sigma \leq s$ ,  $|\gamma| > 0$ , and thus  $|\beta| \leq k-1 \leq s-1$ , where  $(e_1, \dots, e_l)$ ,  $w$  and  $u_m^\epsilon$  denote the coefficients of  $E^{-1}$ ,  $C_j$  and  $\mathbf{u}^\epsilon$  respectively, with  $C_j$  taking a form similar to that of  $B_{ij}$ .  $\tilde{H}_2^\sigma$  is a finite sum of terms of the form

$$(\partial_x^{\alpha_1} e_1) \cdots (\partial_x^{\alpha_l} e_l) (\partial_x^\beta w) (\partial_x^\gamma u_m^\epsilon)$$

with  $|\alpha_1| + \cdots + |\alpha_l| + |\beta| + |\gamma| \leq \sigma + 1 \leq s + 1$ ,  $|\gamma| > 1$ , and thus  $|\beta| \leq \sigma - 1 \leq s - 1$ , where  $(e_1, \dots, e_l)$ ,  $w$  and  $u_m^\epsilon$  denote the coefficients of  $E^{-1}$ ,  $B_{ij}$  and  $\mathbf{u}^\epsilon$  respectively. Hence, we have

$$\|\tilde{H}_1^\sigma\|^2 + \|\tilde{H}_2^\sigma\|^2 \leq C(\mathcal{M}).$$

Similarly, we can show that

$$\begin{aligned} & (\mu + \lambda) \int_0^t \int_{\mathbb{R}^d} E[E^{-1} \nabla \operatorname{div}, \{L_E\}^\sigma](0, \mathbf{u}^\epsilon)^T \mathbf{W}_\sigma^\epsilon dx d\tau \\ & \leq \frac{\mu + \lambda}{2} \int_0^t \|\operatorname{div} \mathbf{W}_\sigma^\epsilon\|^2 d\tau + C(\mathcal{N}) \int_0^t \|\mathbf{W}_\sigma^\epsilon\|^2 d\tau + tC(\mathcal{M}). \end{aligned}$$

Finally, the above estimates (3.19)–(3.21) and the positivity of  $E$  imply (3.15).  $\square$

Next, we derive an estimate for  $\|\operatorname{curl}(r_0 \mathbf{u}^\epsilon)\|_{\sigma-1}$ . Define

$$f^\epsilon(S^\epsilon, \epsilon q^\epsilon) := 1 - \frac{r_0(S^\epsilon)}{r^\epsilon(S^\epsilon, \epsilon q^\epsilon)}. \quad (3.23)$$

Hereafter we denote  $r_0(t) := r_0(S^\epsilon(t))$  and  $f^\epsilon(t) := f^\epsilon(S^\epsilon(t), \epsilon q^\epsilon(t))$  for notational simplicity.

One can factor out  $\epsilon q^\epsilon$  in  $f^\epsilon(t)$ . In fact, using Taylor's expansion, one obtains that there exists a smooth function  $g^\epsilon(t)$ , such that

$$f^\epsilon(t) = \epsilon g^\epsilon(t) := \epsilon g^\epsilon(S^\epsilon(t), \epsilon q^\epsilon(t)), \quad \|g^\epsilon(t)\|_s \leq C(\mathcal{M}_\epsilon(T)). \quad (3.24)$$

Since

$$\partial_t S^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) S^\epsilon = \epsilon^2 \frac{\Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon q^\epsilon)},$$

the equations for  $\mathbf{u}^\epsilon$  are equivalent to

$$\begin{aligned} [\partial_t + (\mathbf{u}^\epsilon \cdot \nabla)](r_0 \mathbf{u}^\epsilon) + \frac{1}{\epsilon} \nabla q^\epsilon = & g^\epsilon \nabla q^\epsilon + (1 - \epsilon g^\epsilon)(\text{curl } \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon \\ & + (1 - \epsilon g^\epsilon) \text{div } \Psi^\epsilon + \epsilon^2 r'_0(S^\epsilon) \mathbf{u}^\epsilon \frac{\Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon q^\epsilon)}. \end{aligned} \quad (3.25)$$

We perform the operator *curl* to the equation (3.25) to obtain that

$$\begin{aligned} & [\partial_t + (\mathbf{u}^\epsilon \cdot \nabla)](\text{curl } (r_0 \mathbf{u}^\epsilon)) \\ = & [\mathbf{u}^\epsilon \cdot \nabla, \text{curl}](r_0 \mathbf{u}^\epsilon) + \text{curl}[(1 - \epsilon g^\epsilon)(\text{curl } \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon] \\ & + \text{curl}[(1 - \epsilon g^\epsilon) \text{div } \Psi^\epsilon] + [\text{curl}, g^\epsilon] \nabla q^\epsilon \\ & + \epsilon^2 \text{curl} \left\{ r'_0(S^\epsilon) \mathbf{u}^\epsilon \frac{\Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon q^\epsilon)} \right\}. \end{aligned} \quad (3.26)$$

**Lemma 3.6.** *There exist constants  $C_0 > 0$ ,  $0 < \xi_3 < \mu$ , a function  $C(\cdot)$  from  $[0, \infty)$  to  $[0, \infty)$  and a sufficiently small constant  $\eta_2 > 0$ , such that for all  $\epsilon \in (0, 1]$  and all  $t \in [0, T]$ ,*

$$\begin{aligned} & \|\{\text{curl } (r_0 \mathbf{u}^\epsilon), \text{curl } \mathbf{H}^\epsilon\}(t)\|_{s-1}^2 + \xi_3 \int_0^t \|\nabla \text{curl } \mathbf{u}^\epsilon\|_{s-1}^2 d\tau \\ & \leq C_0 + tC(\mathcal{N}_\epsilon(T)) + \eta_2 \int_0^t \|\mathbf{u}^\epsilon\|_{s+1}^2 d\tau. \end{aligned} \quad (3.27)$$

*Proof.* Set  $\mathcal{N} := \mathcal{N}_\epsilon(T)$  and  $\omega = \text{curl } (r_0 \mathbf{u}^\epsilon)$ . Taking  $\partial_x^\alpha$  ( $|\alpha| \leq s-1$ ) on (3.26), multiplying the resulting equations by  $\partial_x^\alpha \omega$ , and integrating over  $[0, t] \times \mathbb{R}^d$  with  $t \leq T$ , we obtain

$$\begin{aligned} \frac{1}{2} \|\partial^\alpha \omega(t)\|^2 = & \frac{1}{2} \|\partial^\alpha \omega(0)\|^2 - \int_0^t \langle (\mathbf{u}^\epsilon \cdot \nabla) \partial^\alpha \omega, \partial^\alpha \omega \rangle(\tau) d\tau \\ & + \int_0^t \langle [\mathbf{u}^\epsilon \cdot \nabla, \partial_x^\alpha] \omega, \partial^\alpha \omega \rangle(\tau) d\tau \\ & + \int_0^t \langle \partial^\alpha \{[\text{curl}, g^\epsilon] \nabla q^\epsilon\}, \partial^\alpha \omega \rangle(\tau) d\tau \\ & + \int_0^t \langle \partial^\alpha \{[\mathbf{u}^\epsilon \cdot \nabla, \text{curl}](r_0 \mathbf{u}^\epsilon)\}, \partial^\alpha \omega \rangle(\tau) d\tau \\ & + \int_0^t \left\langle \partial^\alpha \left\{ \epsilon^2 \text{curl} \left\{ r'_0(S^\epsilon) \mathbf{u}^\epsilon \frac{\Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon q^\epsilon)} \right\} \right\}, \partial^\alpha \omega \right\rangle(\tau) d\tau \\ & + \int_0^t \langle \partial^\alpha \{\text{curl}[(1 - \epsilon g^\epsilon)(\text{curl } \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon]\}, \partial^\alpha \omega \rangle(\tau) d\tau \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \langle \partial^\alpha \{ \operatorname{curl} [(1 - \epsilon g^\epsilon) \operatorname{div} \Psi^\epsilon] \}, \partial^\alpha \omega \rangle(\tau) d\tau \\
& = : \frac{1}{2} \|\partial^\alpha \omega(0)\|^2 + \int_0^t \sum_{i=1}^7 I_i(\tau) d\tau.
\end{aligned} \tag{3.28}$$

We have to estimate the terms  $I_i(\tau)$  ( $1 \leq i \leq 7$ ) on the right-hand side of (3.28). Applying partial integrations, we have

$$I_1(\tau) = \int_{\mathbb{R}^d} |\partial^\alpha \omega|^2 \operatorname{div} \mathbf{u}^\epsilon \leq \|\operatorname{div} \mathbf{u}^\epsilon(\tau)\|_{L^\infty} \|\partial^\alpha \omega(\tau)\|^2 \leq C(\mathcal{N}) \|\partial^\alpha \omega(\tau)\|^2, \tag{3.29}$$

while for the term  $I_2(\tau)$ , an application of Cauchy-Schwarz's inequality gives

$$|I_2(\tau)| \leq \|\partial^\alpha \omega\| \|\mathbf{h}_\alpha(\tau)\|, \quad \mathbf{h}_\alpha(\tau) := [\mathbf{u}^\epsilon \cdot \nabla, \partial_x^\alpha] \omega.$$

The commutator  $\mathbf{h}_\alpha$  is a sum of terms  $\partial_x^\beta \mathbf{u}^\epsilon \partial_x^\gamma \omega$  with multi-indices  $\beta$  and  $\gamma$  satisfying  $|\beta| + |\gamma| \leq s$ ,  $|\beta| > 0$ , and  $|\gamma| > 0$ . Thus, the inequality (2.7) with  $\sigma = s - 1 > d/2$  implies that  $\|\mathbf{h}_\alpha(\tau)\| \leq C(\mathcal{N})$ . Hence, we have

$$|I_2(\tau)| \leq C(\mathcal{N}) + \|\partial^\alpha \omega(\tau)\|^2. \tag{3.30}$$

Noting that  $([\operatorname{curl}, g^\epsilon] \mathbf{a})_{i,j} = a_i \partial_{x_j} g^\epsilon - a_j \partial_{x_i} g^\epsilon$  for  $\mathbf{a} = (a_1, \dots, a_d)$ , the inequality (2.7), and the estimate (3.24), we can control the term  $I_3(\tau)$  as follows

$$\begin{aligned}
|I_3(\tau)| & \leq \|\partial^\alpha \{ [\operatorname{curl}, g^\epsilon] \nabla q^\epsilon \}\| \|\partial^\alpha \omega\| \\
& \leq K \|[\operatorname{curl}, g^\epsilon] \nabla q^\epsilon\|_{s-1} \|\partial^\alpha \omega\| \\
& \leq K \|\nabla g^\epsilon(\tau)\|_{s-1} \|\nabla q^\epsilon(\tau)\|_{s-1} \|\partial^\alpha \omega\| \\
& \leq C(\mathcal{N}) + \|\partial^\alpha \omega(\tau)\|^2.
\end{aligned} \tag{3.31}$$

Similarly, the term  $I_4(\tau)$  can be bounded as follows.

$$\begin{aligned}
|I_4(\tau)| & \leq K \|\partial^\alpha \{ [\mathbf{u}^\epsilon \cdot \nabla, \operatorname{curl}] (r_0 \mathbf{u}^\epsilon) \}\| \|\partial^\alpha \omega\| \\
& \leq K \|[\mathbf{u}^\epsilon \cdot \nabla, \operatorname{curl}] (r_0 \mathbf{u}^\epsilon)\|_{s-1} \|\partial^\alpha \omega\| \\
& \leq K \|[\mathbf{u}_j^\epsilon, \operatorname{curl}] \partial_{x_j} (r_0 \mathbf{u}^\epsilon)\|_{s-1} \|\partial^\alpha \omega\| \\
& \leq C(\mathcal{N}) + \|\partial^\alpha \omega(\tau)\|^2.
\end{aligned} \tag{3.32}$$

To bound the term  $I_5(\tau)$ , we use the Moser-type inequality (see [23]) to deduce

$$\begin{aligned}
|I_5(\tau)| & \leq \epsilon^2 K \left\| \partial^\alpha \left\{ \operatorname{curl} \left\{ r_0'(S^\epsilon) \mathbf{u}^\epsilon \frac{\Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon q^\epsilon)} \right\} \right\} \right\| \cdot \|\partial^\alpha \omega\| \\
& = \epsilon^2 K \left\| \partial^\alpha \left[ \left( \frac{r_0'(S^\epsilon) \Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon q^\epsilon)} \right) \operatorname{curl} \mathbf{u}^\epsilon \right] \right\| \cdot \|\partial^\alpha \omega\| \\
& \quad + \epsilon^2 K \left\| \partial^\alpha \left[ \nabla \left( \frac{r_0'(S^\epsilon) \Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon q^\epsilon)} \right) \times \mathbf{u}^\epsilon \right] \right\| \cdot \|\partial^\alpha \omega\| \\
& \leq \epsilon^2 K \|\operatorname{curl} \mathbf{u}^\epsilon\|_{L^\infty} \left\| D^{s-1} \left( \frac{r_0'(S^\epsilon) \Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon q^\epsilon)} \right) \right\| \cdot \|\partial^\alpha \omega\| \\
& \quad + \epsilon^2 K \|D^{s-1}(\operatorname{curl} \mathbf{u}^\epsilon)\| \left\| \frac{r_0'(S^\epsilon) \Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon^2 q^\epsilon)} \right\|_{L^\infty} \cdot \|\partial^\alpha \omega\| \\
& \quad + \epsilon^2 K \|\mathbf{u}^\epsilon\|_{L^\infty} \left\| D^s \left( \frac{r_0'(S^\epsilon) \Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon^2 q^\epsilon)} \right) \right\| \cdot \|\partial^\alpha \omega\|
\end{aligned}$$

$$\begin{aligned}
& + \epsilon^2 K \|D^{s-1} \mathbf{u}^\epsilon\| \left\| \nabla \left( \frac{r'_0(S^\epsilon) \Psi^\epsilon : \nabla \mathbf{u}^\epsilon}{b^\epsilon(S^\epsilon, \epsilon^2 q^\epsilon)} \right) \right\|_{L^\infty} \cdot \|\partial^\alpha \omega\| \\
& \leq C(\mathcal{N}) + \epsilon^2 C(\mathcal{N}) \|\mathbf{u}^\epsilon\|_{s+1}^2 + \|\partial^\alpha \omega(\tau)\|^2,
\end{aligned} \tag{3.33}$$

where the condition  $s > 2 + d/2$  and the inequality (2.8) have been used. For the term  $I_6(\tau)$ , by virtue of (2.1),  $\operatorname{curl} \operatorname{curl} \mathbf{a} = \nabla \operatorname{div} \mathbf{a} - \Delta \mathbf{a}$ . Thus, we integrate by parts to see that

$$\begin{aligned}
I_6(\tau) & = \langle \partial^\alpha \{ (1 - \epsilon g^\epsilon) (\operatorname{curl} \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon \}, \partial^\alpha \{ \operatorname{curl} \operatorname{curl} (r_0 \mathbf{u}^\epsilon) \} \rangle \\
& = \langle \partial^\alpha \{ (1 - \epsilon g^\epsilon) (\operatorname{curl} \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon \}, \partial^\alpha \{ \nabla \operatorname{div} (r_0 \mathbf{u}^\epsilon) - \Delta (r_0 \mathbf{u}^\epsilon) \} \rangle,
\end{aligned}$$

and use Cauchy-Schwarz's inequality and (2.8) to conclude

$$|I_6(\tau)| \leq C(\mathcal{N}) + \theta_1 \|\mathbf{u}^\epsilon(\tau)\|_{s+1}^2, \tag{3.34}$$

where  $\theta_1 > 0$  is a sufficiently small constant independent of  $\epsilon$ . Next, we deal with the term  $I_7(\tau)$ . By the vector identities and integration by parts, we see that there exists a sufficiently small  $\theta_2$ , such that

$$\begin{aligned}
I_7(\tau) & \leq -\inf\{r_0(S^\epsilon)\} \|\nabla \operatorname{curl} \mathbf{u}^\epsilon(\tau)\|_{\sigma-1} + C(\mathcal{N}) \\
& \quad + \theta_2 \|\mathbf{u}^\epsilon(\tau)\|_{s+1} + \epsilon C(\mathcal{N}) \|\mathbf{u}^\epsilon(\tau)\|_{s+1}.
\end{aligned} \tag{3.35}$$

Finally, to estimate  $\|\operatorname{curl} \mathbf{H}^\epsilon\|_{s-1}$ , we apply the operator  $\operatorname{curl}$  to (1.18) and use the vector identity (1.22) to obtain

$$\begin{aligned}
& \partial_t (\operatorname{curl} \mathbf{H}^\epsilon) + \mathbf{u}^\epsilon \cdot \nabla (\operatorname{curl} \mathbf{H}^\epsilon) \\
& = -[\operatorname{curl}, \mathbf{u}^\epsilon] \cdot \nabla \mathbf{H}^\epsilon + \operatorname{curl} ((\mathbf{H}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon - \mathbf{H}^\epsilon \operatorname{div} \mathbf{u}^\epsilon).
\end{aligned} \tag{3.36}$$

By the commutator inequality and Sobolev's inequalities, we find that

$$\|[\operatorname{curl}, \mathbf{u}^\epsilon] \cdot \nabla \mathbf{H}^\epsilon\| \leq C(\mathcal{N})$$

and

$$\|\operatorname{curl} ((\mathbf{H}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon - \mathbf{H}^\epsilon \operatorname{div} \mathbf{u}^\epsilon)\| \leq C(\mathcal{N}) + \theta_3 \|\mathbf{u}^\epsilon\|_{s+1}^2$$

for sufficiently small constant  $\theta_3 > 0$ . Then, by arguments similar to those used in Lemma 3.2, we derive that

$$\|\operatorname{curl} \mathbf{H}^\epsilon\|_{s-1}^2 \leq C_0 + tC(\mathcal{N}) + \theta_3 \int_0^t \|\mathbf{u}^\epsilon(\tau)\|_{s+1}^2 d\tau. \tag{3.37}$$

Thus, the lemma follows from adding up the estimates (3.28)–(3.37) for all  $|\alpha| \leq s-1$  and choosing constants  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  appropriately small.  $\square$

Next we complete the proof of Theorem 3.1 by the following estimate.

**Lemma 3.7.** *There exist constants  $C_0 > 0$ ,  $0 < \xi_4 < \mu$ , and a function  $C(\cdot)$  from  $[0, \infty)$  to  $[0, \infty)$ , such that for all  $\epsilon \in (0, 1]$  and  $t \in [0, T]$ ,*

$$\|(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)(t)\|_s^2 + \xi_4 \int_0^t \|\mathbf{u}^\epsilon\|^2 d\tau \leq C_0 + (t + \epsilon)C(\mathcal{M}_\epsilon(T)). \tag{3.38}$$

*Proof.* First, from (3.14) we get

$$\|\mathbf{u}^\epsilon\|_{\sigma+1}^2 \leq \tilde{C} \left\{ \|\nabla(\{L_{E^\epsilon}(\partial_x)\}^\sigma \mathbf{W}^\epsilon)\|_0^2 + \|\nabla \operatorname{curl} (r_0 \mathbf{u}^\epsilon)\|_{\sigma-1}^2 + \|\mathbf{u}^\epsilon\|_\sigma^2 \right\}, \tag{3.39}$$

where  $\tilde{C} := C_1 + tC(\mathcal{M}_\epsilon(T)) + \epsilon C(\mathcal{M}_\epsilon(T))$ . Moreover, using Lemma 3.2, we obtain

$$\|\mathbf{u}^\epsilon\|_{\sigma+1}^2 \leq \tilde{C} \left\{ \|\nabla\{L_{E^\epsilon}(\partial_x)\}^\sigma \mathbf{W}^\epsilon\|_0^2 + \|\nabla(\operatorname{curl} \mathbf{u}^\epsilon)\|_{\sigma-1}^2 + \|\mathbf{u}^\epsilon\|_\sigma^2 \right\}. \tag{3.40}$$

In view of (3.40), there exists a constant  $\kappa_2$  such that

$$\begin{aligned} \frac{\xi_2}{2} \|\nabla \mathbf{W}_\sigma^\epsilon\|_0^2 + \xi_3 \|\nabla \operatorname{curl} \mathbf{u}^\epsilon\|_{\sigma-1}^2 &\geq \kappa_2 \|\mathbf{u}^\epsilon\|_{\sigma+1}^2 - \tilde{C}_1 \left\{ \|\nabla \{L_{E^\epsilon}(\partial_x)\}^\sigma \mathbf{W}^\epsilon\|_0^2 \right. \\ &\quad \left. + \|\nabla(\operatorname{curl} \mathbf{u}^\epsilon)\|_{\sigma-1}^2 \right\} - \tilde{C} \|\mathbf{u}^\epsilon\|_\sigma^2, \end{aligned} \quad (3.41)$$

where  $\tilde{C}_1 = tC(\mathcal{M}_\epsilon(T)) + \epsilon C(\mathcal{M}_\epsilon(T))$ . Now, we combine the estimates (3.15) and (3.27) with (3.41), and use the fact that  $\operatorname{div} \mathbf{H} = 0$  to conclude that there exists a positive constant  $\kappa_3$ , such that

$$\begin{aligned} &\|(L_{E^\epsilon}(\partial_x))^\sigma \mathbf{U}^\epsilon\|_0^2 + \|\operatorname{curl}(r_0 \mathbf{u}^\epsilon)\|_{\sigma-1}^2 + \kappa_3 \int_0^t \|\mathbf{u}^\epsilon\|_{\sigma+1}^2 dx \\ &\leq C_0 + tC(\mathcal{M}_\epsilon(T)) + \tilde{C} \int_0^t \|\mathbf{u}^\epsilon\|_\sigma^2 d\tau \\ &\quad + \tilde{C}_1 \int_0^t \left\{ \|\nabla \{L_{E^\epsilon}(\partial_x)\}^\sigma \mathbf{W}^\epsilon\|_0^2 + \|\nabla(\operatorname{curl} \mathbf{u}^\epsilon)\|_{\sigma-1}^2 \right\} d\tau \\ &\leq C_0 + (t + \epsilon)C(\mathcal{M}_\epsilon(T)) + \tilde{C} \int_0^t \|\mathbf{u}^\epsilon\|_\sigma^2 d\tau \end{aligned}$$

for sufficiently small  $\eta_1$  and  $\eta_2$ . Thus by induction, we conclude that

$$\begin{aligned} &\|\{L_{E^\epsilon}(\partial_x)\}^\sigma \mathbf{U}^\epsilon\|_0^2 + \|\operatorname{curl}(r_0 \mathbf{u}^\epsilon)\|_{\sigma-1}^2 \\ &\quad + \kappa_3 \int_0^t \|\mathbf{u}^\epsilon\|_{\sigma+1}^2 d\tau \leq C_0 + (t + \epsilon)C(\mathcal{M}_\epsilon(T)). \end{aligned}$$

Using (3.14) again, we obtain the estimate (3.38) by induction on  $\sigma \in \{0, \dots, s\}$ .  $\square$

#### 4. INCOMPRESSIBLE LIMIT

In this section, we shall prove the convergence part of Theorem 1.1 by modifying the method developed by Métivier and Schochet [27], see also some extensions [1, 2, 26].

*Proof of the convergence part of Theorem 1.1.* The uniform bound (1.29) implies that, after extracting a subsequence, one gets the following limits:

$$(q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon) \rightharpoonup (q, \mathbf{v}, \bar{\mathbf{H}}) \quad \text{weakly-}^* \text{ in } L^\infty(0, T; H^s(\mathbb{R}^d)). \quad (4.1)$$

The equations (1.18) and (1.19) imply that  $\partial_t S^\epsilon$  and  $\partial_t \mathbf{H}^\epsilon \in C([0, T], H^{s-1}(\mathbb{R}^d))$ . Thus, after further extracting a subsequence, we obtain that, for all  $s' < s$ ,

$$S^\epsilon \rightarrow \bar{S} \quad \text{strongly in } C([0, T], H_{\operatorname{loc}}^{s'}(\mathbb{R}^d)), \quad (4.2)$$

$$\mathbf{H}^\epsilon \rightarrow \bar{\mathbf{H}} \quad \text{strongly in } C([0, T], H_{\operatorname{loc}}^{s'}(\mathbb{R}^d)), \quad (4.3)$$

where the limit  $\bar{\mathbf{H}} \in C([0, T], H_{\operatorname{loc}}^{s'}(\mathbb{R}^d)) \cap L^\infty(0, T; H_{\operatorname{loc}}^s(\mathbb{R}^d))$ . Similarly, by (3.26) and the uniform bound (1.29), we have

$$\operatorname{curl}(r_0(S^\epsilon) \mathbf{u}^\epsilon) \rightarrow \operatorname{curl}(r_0(\bar{S}) \mathbf{v}) \quad \text{strongly in } C([0, T], H_{\operatorname{loc}}^{s'-1}(\mathbb{R}^d)) \quad (4.4)$$

for all  $s' < s$ , where  $r_0(\bar{S}) = \lim_{\epsilon \rightarrow 0} r_0(S^\epsilon) := \lim_{\epsilon \rightarrow 0} r^\epsilon(S^\epsilon, 0)$ .

In order to obtain the limit system, we need to prove that the convergence in (4.1) holds in the strong topology of  $L^2(0, T; H_{\operatorname{loc}}^{s'}(\mathbb{R}^d))$  for all  $s' < s$ . To this end, we first show that  $q = 0$  and  $\operatorname{div} \mathbf{v} = 0$ . In fact, from (3.16) we get

$$\epsilon E^\epsilon(S^\epsilon, \epsilon q^\epsilon) \partial_t \mathbf{U}^\epsilon + L(\partial_x) \mathbf{U}^\epsilon = -\epsilon E^\epsilon(S^\epsilon, \epsilon q^\epsilon) \mathbf{u}^\epsilon \cdot \nabla \mathbf{U}^\epsilon + \epsilon(\mathbf{J}^\epsilon + \mathbf{V}^\epsilon). \quad (4.5)$$



Since

$$E^\epsilon(S^\epsilon, \epsilon q^\epsilon) - E^\epsilon(S^\epsilon, 0) = O(\epsilon),$$

we have

$$\epsilon E^\epsilon(S^\epsilon, 0) \partial_t \mathbf{U}^\epsilon + L(\partial_x) \mathbf{U}^\epsilon = \epsilon \mathbf{h}^\epsilon, \quad (4.6)$$

where  $\mathbf{h}^\epsilon$  is uniformly bounded in  $C([0, T], H^{s-2}(\mathbb{R}^d))$  in view of (1.29). Passing to the weak limit to (4.6), we obtain  $\nabla q = 0$  and  $\operatorname{div} \mathbf{v} = 0$ . Since  $q \in L^\infty(0, T; H^s(\mathbb{R}^d))$ , we infer that  $q = 0$ . Noticing that the strong compactness for the incompressible components by (4.4), it is sufficient to prove the following proposition on the acoustic components.

**Proposition 4.1.** *Suppose that the assumptions in Theorem 1.1 hold, then  $q^\epsilon$  converges strongly to 0 in  $L^2(0, T; H_{\text{loc}}^{s'}(\mathbb{R}^d))$  for all  $s' < s$ , and  $\operatorname{div} \mathbf{u}^\epsilon$  converges strongly to 0 in  $L^2(0, T; H_{\text{loc}}^{s'-1}(\mathbb{R}^d))$  for all  $s' < s$ .*

The proof of Proposition 4.1 is built on the the following dispersive estimates on the wave equations obtained by Métivier and Schochet [27] and reformulated in [2].

**Lemma 4.2** ([2, 27]). *Let  $T > 0$  and  $w^\epsilon$  be a bounded sequence in  $C([0, T], H^2(\mathbb{R}^d))$ , such that*

$$\epsilon^2 \partial_t (a^\epsilon \partial_t w^\epsilon) - \nabla \cdot (b^\epsilon \nabla w^\epsilon) = c^\epsilon,$$

where  $c^\epsilon$  converges to 0 strongly in  $L^2(0, T; L^2(\mathbb{R}^d))$ . Assume further that, for some  $s > d/2 + 1$ , the coefficients  $(a^\epsilon, b^\epsilon)$  are uniformly bounded in  $C([0, T]; H^s(\mathbb{R}^d))$  and converges in  $C([0, T]; H_{\text{loc}}^s(\mathbb{R}^d))$  to a limit  $(a, b)$  satisfying the decay estimate

$$\begin{aligned} |a(x, t) - \underline{a}| &\leq C_0 |x|^{-1-\delta}, & |\nabla_x a(x, t)| &\leq C_0 |x|^{-2-\delta}, \\ |b(x, t) - \underline{b}| &\leq C_0 |x|^{-1-\delta}, & |\nabla_x b(x, t)| &\leq C_0 |x|^{-2-\delta}, \end{aligned}$$

for some given positive constants  $\underline{a}$ ,  $\underline{b}$ ,  $C_0$  and  $\delta$ . Then the sequence  $w^\epsilon$  converges to 0 in  $L^2(0, T; L_{\text{loc}}^2(\mathbb{R}^d))$ .

*Proof of Proposition 4.1.* We first show that  $q^\epsilon$  converges strongly to 0 in  $L^2(0, T; H_{\text{loc}}^{s'}(\mathbb{R}^d))$  for all  $s' < s$ . An application of the operator  $\epsilon^2 \partial_t$  to (1.16) gives

$$\epsilon^2 \partial_t (a^\epsilon(S^\epsilon, \epsilon q^\epsilon) \partial_t q^\epsilon) + \epsilon \partial_t \operatorname{div} \mathbf{u}^\epsilon = -\epsilon^2 \partial_t \{a^\epsilon(S^\epsilon, \epsilon q^\epsilon) (\mathbf{u}^\epsilon \cdot \nabla) q^\epsilon\}. \quad (4.7)$$

Dividing (1.17) by  $r^\epsilon(S^\epsilon, \epsilon q^\epsilon)$  and then taking the operator  $\operatorname{div}$  to the resulting equation, one has

$$\begin{aligned} \partial_t \operatorname{div} \mathbf{u}^\epsilon + \frac{1}{\epsilon} \operatorname{div} \left( \frac{1}{r^\epsilon(S^\epsilon, \epsilon q^\epsilon)} \nabla q^\epsilon \right) \\ = -\operatorname{div}((\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon) + \operatorname{div} \left( \frac{1}{r^\epsilon(S^\epsilon, \epsilon q^\epsilon)} (\operatorname{curl} \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon \right) \\ + \operatorname{div} \left( \frac{1}{r^\epsilon(S^\epsilon, \epsilon q^\epsilon)} \operatorname{div} \Psi(\mathbf{u}^\epsilon) \right). \end{aligned} \quad (4.8)$$

Subtracting (4.8) from (4.7), we obtain

$$\epsilon^2 \partial_t (a^\epsilon(S^\epsilon, \epsilon q^\epsilon) \partial_t q^\epsilon) - \operatorname{div} \left( \frac{1}{r^\epsilon(S^\epsilon, \epsilon q^\epsilon)} \nabla q^\epsilon \right) = F^\epsilon(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon), \quad (4.9)$$

where

$$F^\epsilon(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon) = \epsilon \operatorname{div} \left( \frac{1}{r^\epsilon(S^\epsilon, \epsilon q^\epsilon)} (\operatorname{curl} \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon \right)$$

$$\begin{aligned}
& + \epsilon \operatorname{div} \left( \frac{1}{r^\epsilon(S^\epsilon, \epsilon q^\epsilon)} \operatorname{div} \Psi(\mathbf{u}^\epsilon) \right) - \epsilon \operatorname{div}((\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon) \\
& - \epsilon^2 \partial_t \{a^\epsilon(S^\epsilon, \epsilon q^\epsilon)(\mathbf{u}^\epsilon \cdot \nabla) q^\epsilon\}.
\end{aligned}$$

In view of the uniform boundedness of  $(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)$ , the smoothness and positivity assumptions on  $a^\epsilon$  and  $r^\epsilon$ , and the convergence of  $S^\epsilon$ , we find that

$$F^\epsilon(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon) \rightarrow 0 \quad \text{strongly in } L^2(0, T; L^2(\mathbb{R}^d)),$$

and the coefficients in (4.9) satisfy the requirements in Lemma 4.2. Therefore, by virtue of Lemma 4.2,

$$q^\epsilon \rightarrow 0 \quad \text{strongly in } L^2(0, T; L^2_{\text{loc}}(\mathbb{R}^d)).$$

On the other hand, the uniform boundedness of  $q^\epsilon$  in  $C([0, T], H^s(\mathbb{R}^d))$  and an interpolation argument yield that

$$q^\epsilon \rightarrow 0 \quad \text{strongly in } L^2(0, T; H^{s'}_{\text{loc}}(\mathbb{R}^d)) \quad \text{for all } s' < s.$$

Similarly, we can obtain the convergence of  $\operatorname{div} u^\epsilon$ .  $\square$

We continue our proof of Theorem 1.1. From Proposition 4.1, we know that

$$\operatorname{div} \mathbf{u}^\epsilon \rightarrow \operatorname{div} \mathbf{v} \quad \text{in } L^2(0, T; H^{s'-1}_{\text{loc}}(\mathbb{R}^d)).$$

Hence, from (4.4) it follows that

$$\mathbf{u}^\epsilon \rightarrow \mathbf{v} \quad \text{in } L^2(0, T; H^{s'}_{\text{loc}}(\mathbb{R}^d)) \quad \text{for all } s' < s.$$

By (4.2), (4.3) and Proposition 4.2, we obtain

$$\begin{aligned}
r^\epsilon(S^\epsilon, \epsilon q^\epsilon) & \rightarrow r_0(\bar{S}) \quad \text{in } L^\infty(0, T; L^\infty(\mathbb{R}^d)); \\
\nabla \mathbf{u}^\epsilon & \rightarrow \nabla \mathbf{v} \quad \text{in } L^2(0, T; H^{s'-1}_{\text{loc}}(\mathbb{R}^d)); \\
\nabla \mathbf{H}^\epsilon & \rightarrow \nabla \bar{\mathbf{H}} \quad \text{in } L^2(0, T; H^{s'-1}_{\text{loc}}(\mathbb{R}^d)).
\end{aligned}$$

Passing to the limit in the equations for  $S^\epsilon$  and  $\mathbf{H}^\epsilon$ , we see that the limits  $\bar{S}$  and  $\bar{\mathbf{H}}$  satisfy

$$\partial_t \bar{S} + (\mathbf{v} \cdot \nabla) \bar{S} = 0, \quad \partial_t \bar{\mathbf{H}} + (\mathbf{v} \cdot \nabla) \bar{\mathbf{H}} - (\bar{\mathbf{H}} \cdot \nabla) \mathbf{v} = 0$$

in the sense of distributions. Since  $r^\epsilon(S^\epsilon, \epsilon q^\epsilon) - r_0(S^\epsilon) = O(\epsilon)$ , we have

$$(r^\epsilon(S^\epsilon, \epsilon q^\epsilon) - r_0(S^\epsilon))(\partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon) \rightarrow 0,$$

whence,

$$\begin{aligned}
r^\epsilon(S^\epsilon, \epsilon q^\epsilon)(\partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon) & = (r^\epsilon(S^\epsilon, \epsilon q^\epsilon) - r_0(S^\epsilon))(\partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon) \\
& \quad + \partial_t(r_0(S^\epsilon) \mathbf{u}^\epsilon) + (\mathbf{u}^\epsilon \cdot \nabla)(r_0(S^\epsilon) \mathbf{u}^\epsilon) \\
& \rightarrow r_0(\bar{S})(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v})
\end{aligned}$$

in the sense of distributions.

Applying the operator *curl* to the momentum equation (1.17) and taking to the limit, we find that

$$\operatorname{curl} (r_0(\bar{S})(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - (\operatorname{curl} \bar{\mathbf{H}}) \times \bar{\mathbf{H}} - \mu \Delta \mathbf{v}) = 0.$$

Therefore, by the fact that  $\operatorname{curl} \nabla = 0$ , the limit  $(\bar{S}, \mathbf{v}, \bar{\mathbf{H}})$  satisfies

$$r(\bar{S}, 0)(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) - (\operatorname{curl} \bar{\mathbf{H}}) \times \bar{\mathbf{H}} - \mu \Delta \mathbf{v} + \nabla \pi = 0, \quad (4.10)$$

$$\partial_t \bar{\mathbf{H}} + (\mathbf{v} \cdot \nabla) \bar{\mathbf{H}} - (\bar{\mathbf{H}} \cdot \nabla) \mathbf{v} = 0, \quad (4.11)$$

$$\partial_t \bar{S} + (\mathbf{v} \cdot \nabla) \bar{S} = 0, \quad (4.12)$$

$$\operatorname{div} \mathbf{v} = 0, \quad \operatorname{div} \bar{\mathbf{H}} = 0 \quad (4.13)$$

for some function  $\pi$ .

If we employ the same arguments as in the proof of Theorem 1.5 in [27], we find that  $(\bar{S}, \mathbf{v}, \bar{\mathbf{H}})$  satisfies the initial conditions (1.31). Moreover, the standard iterative method shows that the system (4.10)–(4.13) with initial data (1.31) has a unique solution  $(S^*, \mathbf{v}^*, \mathbf{H}^*) \in C([0, T], H^s(\mathbb{R}^d))$ . Thus, the uniqueness of solutions to the limit system (4.10)–(4.13) implies that the above convergence results hold for the full sequence of  $(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)$ . Thus, the proof is completed.  $\square$

## 5. COMPRESSIBLE NON-ISENTROPIC MHD EQUATIONS WITH INFINITE REYNOLDS NUMBER

In the study of magnetohydrodynamics, for some local processes in the cosmic system, the effect of the magnetic diffusion will become very important, see [13]. Moreover, when the Reynolds number of a fluid is very high and the temperature changes very slowly, we can ignore the viscosity and the heat conductivity of the fluid in the MHD equations. In such situation, the compressible MHD equations in the non-isentropic case take the form:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \quad (5.1)$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = (\operatorname{curl} \mathbf{H}) \times \mathbf{H}, \quad (5.2)$$

$$\partial_t \mathcal{E} + \operatorname{div}(\mathbf{u}(\mathcal{E}' + p)) = \operatorname{div}[(\mathbf{u} \times \mathbf{H}) \times \mathbf{H} + \nu \mathbf{H} \times (\operatorname{curl} \mathbf{H})], \quad (5.3)$$

$$\partial_t \mathbf{H} - \operatorname{curl}(\mathbf{u} \times \mathbf{H}) = -\operatorname{curl}(\nu \operatorname{curl} \mathbf{H}), \quad \operatorname{div} \mathbf{H} = 0. \quad (5.4)$$

As before here  $\rho$  denotes the density,  $\mathbf{u} \in \mathbb{R}^d$  ( $d = 2, 3$ ) the velocity,  $\mathbf{H} \in \mathbb{R}^d$  the magnetic field;  $\mathcal{E}$  the total energy given by  $\mathcal{E} = \mathcal{E}' + |\mathbf{H}|^2/2$  and  $\mathcal{E}' = \rho(e + |\mathbf{u}|^2/2)$  with  $e$  being the internal energy,  $\rho|\mathbf{u}|^2/2$  the kinetic energy, and  $|\mathbf{H}|^2/2$  the magnetic energy. The equations of state  $p = p(\rho, \theta)$  and  $e = e(\rho, \theta)$  relate the pressure  $p$  and the internal energy  $e$  to the density  $\rho$  and the temperature  $\theta$ . The constant  $\nu > 0$  is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field.

Using the Gibbs relation (1.9) and the identities (1.6) and (1.7), the equation of energy conservation (5.3) can be replaced by

$$\partial_t(\rho S) + \operatorname{div}(\rho S \mathbf{u}) = \frac{\nu}{\theta} |\operatorname{curl} \mathbf{H}|^2, \quad (5.5)$$

where  $S$  denotes the entropy.

As in Section 1, we reconsider the equations of state as functions of  $S$  and  $p$ , i.e.,  $\rho = R(S, p)$  and  $\theta = \Theta(S, p)$  for some positive smooth functions  $R$  and  $\Theta$  defined for all  $S$  and  $p > 0$ , and satisfying  $\partial R / \partial p > 0$ . Then, by utilizing (1.1) together with the constraint  $\operatorname{div} \mathbf{H} = 0$ , the system (5.1), (5.2), (5.4) and (5.5) can be rewritten as

$$A(S, p)(\partial_t p + (\mathbf{u} \cdot \nabla)p) + \operatorname{div} \mathbf{u} = 0, \quad (5.6)$$

$$R(S, p)(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}) + \nabla p = (\operatorname{curl} \mathbf{H}) \times \mathbf{H}, \quad (5.7)$$

$$\partial_t \mathbf{H} - \operatorname{curl}(\mathbf{u} \times \mathbf{H}) = -\operatorname{curl}(\nu \operatorname{curl} \mathbf{H}), \quad \operatorname{div} \mathbf{H} = 0, \quad (5.8)$$

$$R(S, p)\Theta(S, p)(\partial_t S + (\mathbf{u} \cdot \nabla)S) = \nu |\operatorname{curl} \mathbf{H}|^2, \quad (5.9)$$

where  $A(S, p)$  is defined by (1.15). By introducing the dimensionless parameter  $\epsilon$ , and making the following changes of variables:

$$\begin{aligned} p(x, t) &= p^\epsilon(x, \epsilon t), \quad S(x, t) = S^\epsilon(x, \epsilon t), \\ \mathbf{u}(x, t) &= \epsilon \mathbf{u}^\epsilon(x, \epsilon t), \quad \mathbf{H}(x, t) = \epsilon \mathbf{H}^\epsilon(x, \epsilon t), \quad \nu = \epsilon \mu^\epsilon, \end{aligned}$$

and  $p^\epsilon(x, \epsilon t) = \underline{p} e^{\epsilon q^\epsilon(x, \epsilon t)}$  for some positive constant  $\underline{p}$ , the system (5.6)–(5.9) can be rewritten as

$$a^\epsilon(S^\epsilon, \epsilon q^\epsilon)(\partial_t q^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) q^\epsilon) + \frac{1}{\epsilon} \operatorname{div} \mathbf{u}^\epsilon = 0, \quad (5.10)$$

$$r^\epsilon(S^\epsilon, \epsilon q^\epsilon)(\partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon) + \frac{1}{\epsilon} \nabla q^\epsilon = (\operatorname{curl} \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon, \quad (5.11)$$

$$\partial_t \mathbf{H}^\epsilon - \operatorname{curl}(\mathbf{u}^\epsilon \times \mathbf{H}^\epsilon) - \mu^\epsilon \Delta \mathbf{H}^\epsilon = 0, \quad \operatorname{div} \mathbf{H}^\epsilon = 0, \quad (5.12)$$

$$b^\epsilon(S^\epsilon, \epsilon q^\epsilon)(\partial_t S^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) S^\epsilon) = \epsilon^2 \mu^\epsilon |\operatorname{curl} \mathbf{H}^\epsilon|^2, \quad (5.13)$$

where we have used the identity  $\operatorname{curl} \operatorname{curl} \mathbf{H}^\epsilon = \nabla \operatorname{div} \mathbf{H}^\epsilon - \Delta \mathbf{H}^\epsilon$ , the constraint  $\operatorname{div} \mathbf{H}^\epsilon = 0$ , and the abbreviations (1.20) and (1.21).

Formally, we obtain from (5.10) and (5.11) that  $\nabla q^\epsilon \rightarrow 0$  and  $\operatorname{div} \mathbf{u}^\epsilon = 0$  as  $\epsilon \rightarrow 0$ . Applying the operator  $\operatorname{curl}$  to (5.11), using the fact that  $\operatorname{curl} \nabla = 0$ , and letting  $\epsilon \rightarrow 0$ , we obtain

$$\operatorname{curl} (r(\bar{S}, 0)(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - (\operatorname{curl} \bar{\mathbf{H}}) \times \bar{\mathbf{H}}) = 0,$$

where we have assumed that  $(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)$  and  $r^\epsilon(S^\epsilon, \epsilon q^\epsilon)$  converge to  $(\bar{S}, 0, \mathbf{v}, \bar{\mathbf{H}})$  and  $r(\bar{S}, 0)$  in some sense, respectively. Finally, Letting  $\mu^\epsilon \rightarrow \mu > 0$  and applying the identity (1.22), we expect to get the following incompressible MHD equations:

$$r(\bar{S}, 0)(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) - (\operatorname{curl} \bar{\mathbf{H}}) \times \bar{\mathbf{H}} + \nabla \hat{\pi} = 0, \quad (5.14)$$

$$\partial_t \bar{\mathbf{H}} + (\mathbf{v} \cdot \nabla) \bar{\mathbf{H}} - (\bar{\mathbf{H}} \cdot \nabla) \mathbf{v} - \mu \Delta \bar{\mathbf{H}} = 0, \quad (5.15)$$

$$\partial_t \bar{S} + (\mathbf{v} \cdot \nabla) \bar{S} = 0, \quad (5.16)$$

$$\operatorname{div} \mathbf{v} = 0, \quad \operatorname{div} \bar{\mathbf{H}} = 0 \quad (5.17)$$

for some function  $\hat{\pi}$ .

We supplement the system (1.16)–(1.19) with initial conditions

$$(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)|_{t=0} = (S_0^\epsilon, q_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{H}_0^\epsilon). \quad (5.18)$$

The main result of this section reads as follows.

**Theorem 5.1.** *Let  $s > d/2 + 2$  be an integer and  $\mu^\epsilon \rightarrow \mu > 0$ . For any constant  $M_0 > 0$ , there is a positive constant  $T = T(M_0)$ , such that for all  $\epsilon \in (0, 1]$  and any initial data  $(S_0^\epsilon, q_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{H}_0^\epsilon)$  satisfying*

$$\|(S_0^\epsilon, q_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{H}_0^\epsilon)\|_{H^s(\mathbb{R}^d)} \leq M_0, \quad (5.19)$$

*the Cauchy problem (5.10)–(5.13), (5.18) has a unique solution  $(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon) \in C^0([0, T], H^s(\mathbb{R}^d))$ , satisfying that for all  $\epsilon \in (0, 1]$  and  $t \in [0, T]$ ,*

$$\|(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)(t)\|_{H^s(\mathbb{R}^d)} \leq N \quad \text{for some constant } N = N(M_0) > 0. \quad (5.20)$$

*Moreover, suppose further that the initial data  $(S_0^\epsilon, q_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{H}_0^\epsilon)$  converge strongly in  $H^s(\mathbb{R}^d)$  to  $(S_0, 0, \mathbf{v}_0, \mathbf{H}_0)$  and  $S_0^\epsilon$  decays sufficiently rapidly at infinity in the sense that*

$$|S_0^\epsilon(x) - \underline{S}| \leq N_0 |x|^{-1-\iota}, \quad |\nabla S_0^\epsilon(x)| \leq N_0 |x|^{-2-\iota} \quad (5.21)$$

for all  $\epsilon \in (0, 1]$  and some positive constants  $\underline{S}$ ,  $N_0$  and  $\iota$ . Then,  $(S^\epsilon, q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)$  converges weakly in  $L^\infty(0, T; H^s(\mathbb{R}^d))$  and strongly in  $L^2(0, T; H_{\text{loc}}^{s'}(\mathbb{R}^d))$  to a limit  $(\bar{S}, 0, \mathbf{v}, \bar{\mathbf{H}})$  for all  $s' < s$ . Moreover,  $(\bar{S}, \mathbf{v}, \bar{\mathbf{H}})$  is the unique solution in  $C([0, T], H^s(\mathbb{R}^d))$  to the system (5.14)–(5.17) with initial data  $(\bar{S}, \mathbf{v}, \bar{\mathbf{H}})|_{t=0} = (S_0, \mathbf{w}_0, \mathbf{H}_0)$ , where  $\mathbf{w}_0 \in H^s(\mathbb{R}^d)$  is determined by

$$\operatorname{div} \mathbf{w}_0 = 0, \quad \operatorname{curl} (r(S_0, 0) \mathbf{w}_0) = \operatorname{curl} (r(S_0, 0) \mathbf{v}_0), \quad r(S_0, 0) := \lim_{\epsilon \rightarrow 0} r^\epsilon(S_0^\epsilon, 0). \quad (5.22)$$

The function  $\hat{\pi} \in C([0, T] \times \mathbb{R}^d)$  satisfies  $\nabla \hat{\pi} \in C([0, T], H^{s-1}(\mathbb{R}^d))$ .

*Sketch of the proof of Theorem 5.1.* As explained before, the main step is to establish the uniform estimate (5.20). For this purpose, we define  $\mathcal{M}_\epsilon(T)$  as follows

$$\mathcal{M}_\epsilon(T) := \mathcal{N}_\epsilon(T)^2 + \int_0^T \|\mathbf{H}^\epsilon\|_{s+1}^2 d\tau, \quad (5.23)$$

where  $\mathcal{N}_\epsilon(T)$  is defined by (3.3). By arguments similar to those used in the proof of Theorem 3.1, one can obtain the desired estimate. Indeed, the arguments are easier since one can use the magnetic diffusion term to control the terms involving  $\mathbf{H}$  in the momentum equations, and therefore we omit the details here.  $\square$

**Acknowledgements:** The authors would like to thank Prof. Fanghua Lin for suggesting this problem and for helpful discussions. This work was partially done when Li visited the Institute of Applied Physics and Computational Mathematics in Beijing. He would like to thank the institute for hospitality. Jiang was supported by the National Basic Research Program under the Grant 2011CB309705 and NSFC (Grant No. 40890154). Ju was supported by NSFC (Grant No. 40890154, 11171035). Li was supported by NSFC (Grant No. 10971094), PAPD, and the Fundamental Research Funds for the Central Universities.

## REFERENCES

- [1] T. Alazard, Incompressible limit of the nonisentropic Euler equations with solid wall boundary conditions, *Adv. Differential Equations*, 10 (2005), 19-44.
- [2] T. Alazard, Low Mach number limit of the full Navier-Stokes equations, *Arch. Ration. Mech. Anal.* 180 (2006), 1-73.
- [3] K. Asano, On the incompressible limit of the compressible Euler equations, *Japan J. Appl. Math.*, 4 (1987), 455-488.
- [4] D. Bresch, B. Desjardins and E. Grenier, Oscillatory Limit with Changing Eigenvalues: A Formal Study. In: *New Directions in Mathematical Fluid Mechanics*, A. V. Fursikov, G. P. Galdi, V. V. Pukhnachev (eds.), 91-105, Birkhäuser Verlag, Basel, 2010.
- [5] D. Bresch, B. Desjardins, E. Grenier and C.-K. Lin, Low Mach number limit of viscous polytropic flows: formal asymptotics in the periodic case, *Stud. Appl. Math.* 109 (2002), 125-149.
- [6] R. Danchin, Low Mach number limit for viscous compressible flows, *M2AN Math. Model. Numer. Anal.* 39 (2005), 459-475.
- [7] B. Desjardins and E. Grenier, Low Mach number limit of viscous compressible flows in the whole space, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 455 (1999), 2271-2279.
- [8] B. Desjardins, E. Grenier, P.-L. Lions and N. Masmoudi, Incompressible limit for solutions of the isentropic Navier-Stokes equations with Dirichlet boundary conditions, *J. Math. Pures Appl.* (9) 78 (1999), 461-471.
- [9] E. Feireisl and A. Novotný, *Singular Limits in Thermodynamics of Viscous Fluids*, Birkhäuser Basel, 2009.
- [10] J.P. Freidberg, Ideal magnetohydrodynamic theory of magnetic fusion systems, *Rev. Modern Phys.* 54 (1982), 801-903.

- [11] I. Gallagher, A remark on smooth solutions of the weakly compressible periodic Navier-Stokes equations, *J. Math. Kyoto Univ.* 40 (2000), 525-540.
- [12] L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer-Verlag, 1997.
- [13] W.R. Hu, *Cosmic Magnetohydrodynamics*, Science Press, Beijing, 1987 (in Chinese).
- [14] X.P. Hu and D.H. Wang, Low Mach number limit of viscous compressible magnetohydrodynamic flows, *SIAM J. Math. Anal.* 41 (2009), 1272-1294.
- [15] T. Iguchi, The incompressible limit and the initial layer of the compressible Euler equation in  $R_n^+$ , *Math. Methods Appl. Sci.* 20 (1997), 945-958.
- [16] H. Isozaki, Singular limits for the compressible Euler equation in an exterior domain, *J. Reine Angew. Math.* 381 (1987), 1-36.
- [17] H. Isozaki, Singular limits for the compressible Euler equation in an exterior domain, I Bodies in an uniform flow, *Osaka J. Math.* 26 (1989), 399-410.
- [18] S. Jiang, Q.C. Ju and F.C. Li, Incompressible limit of the compressible Magnetohydrodynamic equations with periodic boundary conditions, *Comm. Math. Phys.* 297 (2010), 371-400.
- [19] S. Jiang, Q.C. Ju and F.C. Li, Incompressible limit of the compressible magnetohydrodynamic equations with vanishing viscosity coefficients, *SIAM J. Math. Anal.* 42 (2010), 2539-2553.
- [20] S. Jiang, Q.C. Ju and F.C. Li, Low Mach number limit for the multi-dimensional full magnetohydrodynamic equations, Preprint, available at arXiv:1105.0729v1 [math.AP].
- [21] S. Jiang, Q.C. Ju, F.C. Li and Z.P. Xin, Low Mach number limit for the full compressible magnetohydrodynamic equations with general initial data, submitted, 2011.
- [22] S. Jiang and Y.B. Ou, Incompressible limit of the non-isentropic Navier-Stokes equations with well-prepared initial data in three-dimensional bounded domains, *J. Math. Pures Appl.* 96 (2011), 1-28.
- [23] S. Klainerman and A. Majda, Singular perturbations of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, *Comm. Pure Appl. Math.* 34 (1981), 481-524.
- [24] P. Kukučka, Singular Limits of the Equations of Magnetohydrodynamics, *J. Math. Fluid Mech.* 13 (2011), 173-189.
- [25] Y.-S. Kwon and K. Trivisa, On the incompressible limits for the full magnetohydrodynamics flows, *J. Differential Equations* 251 (2011), 1990-2023.
- [26] C.D. Levermore, W. Sun and K. Trivisa, A low Mach number limit of a dispersive Navier-Stokes system, Preprint, 2011.
- [27] G. Métivier and S. Schochet, The incompressible limit of the non-isentropic Euler equations, *Arch. Ration. Mech. Anal.* 158 (2001), 61-90.
- [28] G. Métivier and S. Schochet, Averaging theorems for conservative systems and the weakly compressible Euler equations, *J. Differential Equations* 187 (2003), 106-183.
- [29] A. Novotný, M. Ruzicka and G. Thäter, Singular limit of the equations of magnetohydrodynamics in the presence of strong stratification, *Math. Models Methods Appl. Sci.* 21 (2011), 115-147.
- [30] P.-L. Lions and N. Masmoudi, Incompressible limit for a viscous compressible fluid, *J. Math. Pures Appl.* 77 (1998), 585-627.
- [31] N. Masmoudi, Examples of singular limits in hydrodynamics, In: *Handbook of Differential Equations: Evolutionary equations*, Vol. III, 195-275, Elsevier/North-Holland, Amsterdam, 2007.
- [32] S. Schochet, Fast singular limits of hyperbolic PDEs, *J. Differential Equations* 114 (1994), 476-512.
- [33] S. Schochet, The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit, *Comm. Math. Phys.* 104 (1986), 49-75.
- [34] S. Schochet, The mathematical theory of the incompressible limit in fluid dynamics, in: *Handbook of Mathematical Fluid Dynamics*, Vol. IV, pp.123-157, Elsevier/North-Holland, Amsterdam, 2007.
- [35] S. Ukai, The incompressible limit and the initial layer of the compressible Euler equation, *J. Math. Kyoto Univ.*, 26 (1986), 323-331.
- [36] A.I. Vol'pert and S.I. Khudjaev, On the Cauchy problem for composite systems of nonlinear equations, *Mat. Sbornik* 87 (1972), 504-528.

LCP, INSTITUTE OF APPLIED PHYSICS AND COMPUTATIONAL MATHEMATICS, P.O. Box 8009,  
BEIJING 100088, P.R. CHINA  
*E-mail address:* `jiang@iapcm.ac.cn`

INSTITUTE OF APPLIED PHYSICS AND COMPUTATIONAL MATHEMATICS, P.O. Box 8009-28, BEI-  
JING 100088, P.R. CHINA  
*E-mail address:* `qiangchang_ju@yahoo.com`

DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, P.R. CHINA  
*E-mail address:* `fli@nju.edu.cn`